On the Application of Homotopy Analysis Method to Fractional Differential Equations

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Abstract: In this paper, an attempt has been made to obtain the solution of linear and nonlinear fractional differential equations by applying an analytic technique, namely the homotopy analysis method (HAM). The fractional derivatives are described by Caputo's sense. By this method, the solution considered as the sum of an infinite series, which converges rapidly to exact solution with the help of the nonzero convergence control parameter $\hbar$. Some examples are given to show the efficiency and accuracy of this method. The solutions obtained by this method have been compared with exact solution. Also our graphical representation of the solutions have been given by using MATLAB software.

Keywords: Fractional differential equations, Caputo fractional derivative, Homotopy analysis method.

Introduction:

In the past years decade's fractional calculus starts to intervene significantly in engineering, physics, biology and economics etc (Kilbas and Srivastava,2006)., Fractional calculus was introduced on September 30, 1695 (Miller and Ross,1993).Several methods have been used to obtain approximate solutions of fractional differential equations, such as the Adomian's decomposition method, (Momani and Shawagfeh,2006). He's variational iteration method (Inc,2008), homotopy perturbation method (Odibat and Momani,2008), spectral methods (Hashim et al., 2009) , and other methods (Yuste,2006). The homotopy analysis method (HAM) proposed in 1992 by Shijun Liao (Al-Rabtah et al., 2004), which is an analytic approximation method with guarantee of convergence, mainly for nonlinear and linear

differential equations. This method has been successfully applied to solve many types of nonlinear differential equations (Wang, and Liao,2005).

The aims of the present paper is to discourse the homotopy analysis method (HAM) and show how it can be used to solve linear and nonlinear fractional differential equations.

After this section 1, introduction, this paper is organized as follows. In section 2, some basic definitions and properties of fractional calculus. In section 3, the HAM is analyzed. In section 4, we apply HAM for solving fractional differential equation. Conclusion is presented in section 5.

1 Preliminaries

In this section, we gives some basic definitions and properties of the Fractional calculus.

Definition 1 A real function $f(x)$, $x > 0$, is said to be in the space C_μ , $\mu \in \mathbb{R}$, if there exists a real number $p > \mu$, such that $f(x) = x^p f(x)$, where $f(x) \in C[a, b]$, and it is said to be in the space C_μ^n , if and only if $f^{(n)} \in C_\mu$, $n \in \mathbb{N}$.

Definition 2 The Riemann-Liouville fractional integral operator (I^α) of order $\alpha \geq 0$, of a function $f(x) \in C_\mu$, $\mu \in \mathbb{N}$. (Ford and Simpson,2003)

$$\left\{ \begin{array}{l} I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^t (x - \tau)^{\alpha-1} f(\tau) d\tau \\ I^0 f(x) = f(x) \end{array} \right\} \quad (1)$$

Properties of the operator I^α can be found in [28-29], we mention only the following

$$I^\alpha I^\beta f(x) = I^{\alpha+\beta} f(x) = I^\beta I^\alpha f(x) \quad (2)$$

$$I^\alpha x^v = \frac{\Gamma(v+1)}{\Gamma(v+\alpha+1)} x^{v+\alpha} \quad (3)$$

For $f(x) \in C_\mu$, $\mu \in \mathbb{N}$, $\alpha, \beta \geq 0$ and $v \geq -1$.

Definition 3 Suppose that $\alpha > 0$, $x > 0$, $\alpha, x \in \mathbb{R}$, $n \in \mathbb{N}$, $n - 1 < \alpha \leq n$. The Caputo fractional differential operator of order α define as (Ford and Simpson,2003)

$$D_c^\alpha f(x) = \left\{ \begin{array}{l} \frac{1}{\Gamma(n-\alpha)} \int_0^t (x - \tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau \\ \frac{d^n}{dx^n} f(x) \end{array} \right. \quad \alpha = n \in \mathbb{N} \quad (4)$$

According to the Caputo's derivative, we can obtain

$$D_c^\alpha C = 0, \quad \text{where } C \text{ is constant,}$$

$$D_c^\alpha x^v = \frac{\Gamma(v+1)}{\Gamma(v-\alpha+1)} x^{v-\alpha} \quad (5)$$

Definition 4 The generalized Mittag-Leffler function is defined by the power series (Ford and Simpson,2005)

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \quad E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha > 0, \beta \in \mathbb{R} \quad (6)$$

2 Homotopy Analysis Method (HAM)

To describe the basic ideas of the HAM (Al-Rabtah *et al.*, ,2012), we consider the following differential equation

$$N[y(t)] = 0, \quad t > 0 \quad (7)$$

where N is a nonlinear operator, and $y(t)$ is unknown function of the variable t .

Based on the zero-order deformation equation constructed by Liao (Al-Rabtah and others ,2012), we give the following zero-order deformation equation in the similar way:

$$(1-q)\ell[\varphi(t;q) - y_0(t)] = q\hbar H(t)[N\varphi(t;q)], \quad (8)$$

where $q \in [0,1]$ is an embedding parameter, $\hbar$ are non-zero auxiliary parameter for $H(t)$ are non-zero auxiliary function, N is nonlinear differential operator, $\varphi(t;q)$ is an unknown function, and $y_0(t)$ is an initial guess of $y(t)$, ℓ is an auxiliary linear integral order operator and it possesses the property $\ell(c) = 0$.

when $q = 0$ and $q = 1$, we have

$$\varphi(t;0) = y_0(t) \quad , \quad \varphi(t;1) = y(t) \quad (9)$$

Expanding φ in Taylor series with respect to q , one has

$$\varphi(t;q) = y_0(t) + \sum_{m=1}^{\infty} y_m(t) \quad (10)$$

where

$$y_m(t) = \frac{1}{m!} \left. \frac{\partial^m \varphi(t;q)}{\partial q^m} \right|_{q=0}, \quad i = 1,2,3, \dots, n \quad (10)$$

Differentiating equation (8) m -times with respect to embedding parameter, then setting $q = 0$, and dividing them by $m!$, we have, using (10), the so-called m th-order deformation equation

$$y_m(t) = \chi_m y_{m-1}(t) + \hbar I^\alpha [H(t) R_m(\overrightarrow{y_{m-1}}(t))] \quad (11)$$

$$R_m(\overrightarrow{y_{m-1}}(t)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} [N\varphi(t; q)]}{\partial q^{m-1}} \Big|_{q=0} \quad (12)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases} \quad (13)$$

here

$$\vec{y}_{m-1} = \{y_0, y_1, y_2, \dots, y_{m-1}\}.$$

Applications

In this section, we apply the homotopy analysis method to find out series solutions for some fractional differential equations.

Example 1. Firstly, we consider the following nonlinear initial value problem (Liao, 2012)

$$D^\alpha y(x) = \frac{40320}{\Gamma(9-\alpha)} x^{8-\alpha} - 3 \frac{\Gamma\left(5 + \frac{\alpha}{2}\right)}{\Gamma\left(5 - \frac{\alpha}{2}\right)} x^{4-\frac{\alpha}{2}} + \frac{9}{4} \Gamma(\alpha+1) + \left(\frac{3}{2} x^{\frac{\alpha}{2}} - x^4\right)^3 - [y(x)]^{\frac{3}{2}}, \quad (14)$$

with

$$y(0) = y'(0) = 0, \quad 0 < \alpha < 2 \quad (15)$$

As before, the second initial condition is for $\alpha > 1$ only. Hence, according to (12), we have

$$R_m(\overrightarrow{y_{m-1}}(t)) = \left[D^\alpha y(x) - \frac{40320}{\Gamma(9-\alpha)} x^{8-\alpha} + 3 \frac{\Gamma\left(5 + \frac{\alpha}{2}\right)}{\Gamma\left(5 - \frac{\alpha}{2}\right)} x^{4-\frac{\alpha}{2}} - \frac{9}{4} \Gamma(\alpha+1) - \left(\frac{3}{2} x^{\frac{\alpha}{2}} - x^4\right)^3 + [y(x)]^{\frac{3}{2}} \right] (1 - \chi_m),$$

In view of the modified of homotopy analysis method, if we set

$$y_0(x) = \left(\frac{3}{2} x^{\frac{\alpha}{2}} - x^4\right)^2, \quad H(x) = 1,$$

then the m th-order deformation equation (11) gives

$$y_1(x) = y_2(x) = \dots = y_n(x) = 0, \quad n > 2$$

then, we get the analytical solution of (14) by HAM as follows:

$$y(x) = y_0(x) + \sum_{m=1}^{\infty} y_m(x) = \left(\frac{3}{2} x^{\frac{\alpha}{2}} - x^4 \right)^2.$$

Thus, $y(x) = \left(\frac{3}{2} x^{\frac{\alpha}{2}} - x^4 \right)^2.$

which is the exact solution of (14).

Figure (1): Shows the exact solution for equation (14), in case ($\alpha = 1$)

Figure (2): Shows the exact solution of equation (14), with $0 < \alpha \leq 1$.

Figure (3): Shows the exact solution of equation (14), with $1 < \alpha \leq 2$.

Example 2. Consider the following linear initial value problem (Hashim, and Momani,2009)

$$\begin{aligned} D^\alpha y(x) + y(x) &= 0, \\ y(0) &= 1, \quad y'(0) = 0. \end{aligned} \tag{15}$$

First, if we set $\alpha = 2$ then the equation (15) becomes linear differential equation of second order which has the following exact solution

$$y(x) = \cos x \tag{16}$$

Since, $N_\alpha[y(x)] = D^\alpha y(x) + y(x)$, according to (11) and (12), we have

$$R_m(\overrightarrow{y_{m-1}}(x)) = D^\alpha y_{m-1}(x) + y_{m-1}(x) . \tag{17}$$

If we take the auxiliary function $H(x) = 1$, we obtain the m^{th} order deformation equation

$$y_m(x) = \chi_m y_{m-1}(x) + \hbar I^\alpha [R_m(\overrightarrow{y_{m-1}}(x))], \tag{18}$$

subject to the initial conditions

$$y_m(0) = 1, \quad y'_m(0) = 0, \tag{19}$$

If we choose the initial guess approximation

$$y_0(x) = 1,$$

For $m = 1$, equation (18) becomes

Then we have

$$\begin{aligned} y_1(x) &= \frac{\hbar}{\Gamma(\alpha + 1)} x^\alpha, \\ y_2(x) &= \frac{(\hbar + \hbar^2)}{\Gamma(\alpha + 1)} x^\alpha + \frac{\hbar^2}{\Gamma(2\alpha + 1)} x^{2\alpha}, \\ y_3(x) &= \frac{(\hbar + 2\hbar^2 + \hbar^3)}{\Gamma(\alpha + 1)} x^\alpha + \frac{2(\hbar^2 + \hbar^3)}{\Gamma(2\alpha + 1)} x^{2\alpha} + \frac{\hbar^3}{\Gamma(3\alpha + 1)} x^{3\alpha} \end{aligned}$$

and so on.

Finally,

we

have

$$y(x) = y_0(x) + \sum_{m=1}^{\infty} y_m(x),$$

$$\begin{aligned}
y(x) = 1 + \frac{\hbar x^\alpha}{\Gamma(\alpha+1)} + \frac{(\hbar + \hbar^2)x^\alpha}{\Gamma(\alpha+1)} + \frac{\hbar^2 x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{(\hbar + 2\hbar^2 + \hbar^3)x^\alpha}{\Gamma(\alpha+1)} \\
+ \frac{2(\hbar^2 + \hbar^3)x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\hbar^3 x^{3\alpha}}{\Gamma(3\alpha+1)} + \dots \quad (20)
\end{aligned}$$

equations (15) is solutions of fractional differential equation (20), taking $\hbar = -1$, we get from equations (20),

$$y(x) = 1 - \frac{1}{\Gamma(\alpha+1)} x^\alpha + \frac{1}{\Gamma(2\alpha+1)} x^{2\alpha} - \frac{1}{\Gamma(3\alpha+1)} x^{3\alpha} + \dots \quad (21)$$

Thus, we have the analytical solution of this problem can be obtained as:

$$y(x) = \sum_{k=0}^{\infty} \frac{(-x^\alpha)^k}{\Gamma(\alpha k + 1)} = E_\alpha(-x^\alpha).$$

Figure (4): Shows the HAM solution (3term approximation) and exact solution of equation (15), at $\alpha = 2$.

Figure (5): Shows the exact solution of equation (15), for different values of α .

Figure (6): Shows the exact solution of equation (15), for different values of α .

Example 3. Consider the following one dimensional fractional heat-like equation:

$$D_t^\alpha u(x, t) = \frac{1}{2} x^2 u_{xx}(x, t), \quad 0 < x < 1, 0 < \alpha \leq 1, t > 0 \quad (22)$$

with initial condition

$$u(x, 0) = x^2$$

According to (11) and (12), the m th-order deformation equation gives by

$$u_m(x, t) = \chi_m u_{m-1}(x, t) + \hbar I^\alpha \left[D_t^\alpha u_{m-1}(x, t) - \frac{1}{2} x^2 u_{xx_{m-1}}(x, t) \right],$$

we can choose the initial guess to be $u_0(x, t) = x^2$,

Consequently, the first few terms of HAM series solutions are as follows:

$$\begin{aligned} u_1(x, t) &= -\hbar \frac{x^2}{\Gamma(\alpha + 1)} t^\alpha, \\ u_2(x, t) &= -(\hbar + \hbar^2) \frac{x^2}{\Gamma(\alpha + 1)} t^\alpha + \hbar^2 \frac{x^2}{\Gamma(2\alpha + 1)} t^{2\alpha}, \\ u_3(x, t) &= -(\hbar + 2\hbar^2 + \hbar^3) \frac{x^2}{\Gamma(\alpha + 1)} t^\alpha + (2\hbar^2 + 2\hbar^3) \frac{x^2}{\Gamma(2\alpha + 1)} t^{2\alpha} \\ &\quad - \hbar^3 \frac{x^2}{\Gamma(3\alpha + 1)} t^{3\alpha}, \end{aligned}$$

and so on. Hence, the HAM series solution is

$$\begin{aligned} u(x, t) &= u_0 + u_1 + u_2 + u_3 + \dots, \\ u(x, t) &= x^2 - \hbar \frac{x^2}{\Gamma(\alpha + 1)} t^\alpha - (\hbar + \hbar^2) \frac{x^2}{\Gamma(\alpha + 1)} t^\alpha + \hbar^2 \frac{x^2}{\Gamma(2\alpha + 1)} t^{2\alpha} \\ &\quad - (\hbar + 2\hbar^2 + \hbar^3) \frac{x^2}{\Gamma(\alpha + 1)} t^\alpha + (2\hbar^2 + 2\hbar^3) \frac{x^2}{\Gamma(2\alpha + 1)} t^{2\alpha} \\ &\quad - \hbar^3 \frac{x^2}{\Gamma(3\alpha + 1)} t^{3\alpha} + \dots. \end{aligned}$$

Taking $\hbar = -1$, we obtain the exact solution,

$$u(x, t) = x^2 \left[1 + \frac{1}{\Gamma(\alpha + 1)} t^\alpha + \frac{1}{\Gamma(2\alpha + 1)} t^{2\alpha} + \frac{1}{\Gamma(3\alpha + 1)} t^{3\alpha} + \dots \right] = x^2 E_\alpha(t^\alpha).$$

Figure (7): Shows the exact solution of equation (22), at $\alpha = 1$.

Figure (8): Shows the HAM solution (3term approximation) of equation (22), at $\alpha = 1$

Figure (9): Shows the exact solution of $u(x, t)$ for equation (22), in case (a) $t = 0$ (b) $x = 1$, for different values of α .

Example 4. Finally, we consider the following fractional order heat equation in 2 dimensional space (Hemeda, 2012)

$$D_t^\alpha u(x, y, t) = b(u_{xx} + u_{yy}), \quad 0 < \alpha \leq 1 \quad (23)$$

with initial condition

$$u(x, y, 0) = c \left(\sin \frac{\pi x}{l} + \sin \frac{\pi y}{l} \right),$$

Since, $N_\alpha[u(x, y, t)] = D_t^\alpha u(x, y, t) - b(u_{xx} + u_{yy})$, according to (11) and (12), we obtain the m th-order deformation equation after chosen the auxiliary linear operator $\ell = D_t^\alpha$, $\alpha > 0$, and the auxiliary function $H(x, y, t) = 1$, we have

$$\begin{aligned} u_m(x, y, t) &= \chi_m u_m(x, y, t) \\ &+ \hbar I^\alpha [R_m(\overline{u_{m-1}})], \end{aligned} \quad (24)$$

$$\text{where } R_m(\overline{y_{m-1}}(t)) = D_t^\alpha u_{m-1}(x, y, t) - b \left(\frac{\partial^2 u_{m-1}}{\partial x^2} + \frac{\partial^2 u_{m-1}}{\partial y^2} \right),$$

we start with the initial approximation

$$u_0(x, y, t) = c \left(\sin \frac{\pi x}{l} + \sin \frac{\pi y}{l} \right),$$

Then according to (24) we find

$$u_1(x, y, t) = c \hbar \frac{b \pi^2}{l^2} \left(\sin \frac{\pi x}{l} + \sin \frac{\pi y}{l} \right) \frac{t^\alpha}{\Gamma(\alpha + 1)},$$

$$\begin{aligned}
u_2(x, y, t) &= c(\hbar + \hbar^2) \frac{b\pi^2}{l^2} \left(\sin \frac{\pi x}{l} + \sin \frac{\pi y}{l} \right) \frac{t^\alpha}{\Gamma(\alpha + 1)} \\
&\quad + c\hbar^2 \frac{b^2\pi^4}{l^4} \left(\sin \frac{\pi x}{l} + \sin \frac{\pi y}{l} \right) \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)}, \\
u_3(x, y, t) &= c(\hbar + 2\hbar^2 + \hbar^3) \frac{b\pi^2}{l^2} \left(\sin \frac{\pi x}{l} + \sin \frac{\pi y}{l} \right) \frac{t^\alpha}{\Gamma(\alpha + 1)} \\
&\quad + c(2\hbar^2 + 2\hbar^3) \frac{b^2\pi^4}{l^4} \left(\sin \frac{\pi x}{l} + \sin \frac{\pi y}{l} \right) \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} \\
&\quad + c\hbar^3 \frac{b^3\pi^6}{l^6} \left(\sin \frac{\pi x}{l} + \sin \frac{\pi y}{l} \right) \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)}.
\end{aligned}$$

and so on. Hence, the HAM series solution is

$$u(x, y, t) = u_0(x, y, t) + \sum_{m=1}^{\infty} u_m(x, y, t),$$

Therefore the approximate solution is given by at $\hbar = -1$.

$$u(x, y, t) = c \left(\sin \frac{\pi x}{l} + \sin \frac{\pi y}{l} \right) \left[1 - \frac{\left(\frac{b\pi^2}{l^2} t^\alpha \right)}{\Gamma(\alpha + 1)} + \frac{\left(\frac{b\pi^2}{l^2} t^\alpha \right)^2}{\Gamma(2\alpha + 1)} - \frac{\left(\frac{b\pi^2}{l^2} t^\alpha \right)^3}{\Gamma(3\alpha + 1)} + \dots \right],$$

Thus,

$$u(x, y, t) = c \left(\sin \frac{\pi x}{l} + \sin \frac{\pi y}{l} \right) E_\alpha \left(-\frac{b\pi^2}{l^2} t^\alpha \right).$$

The result is same as obtained by A. A. Hemeda at (Hemeda,2013)

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Figure (10): The surface shows the analytical solution $u(x, y, t)$ for equation (23), for $t = 0, l = \pi, c = 1$.

Figure (11): The surface shows the analytical solution $u(x, y, t)$ for equation (23), for $x = 0, l = \pi, c = 1, b = -1, \alpha = 2$.

Figure (12): The surface shows the analytical solution $u(x, y, t)$ for equation (23), for $x = 0, l = \pi, c = 1, b = -1, \alpha = 1$.

3 Conclusion

In this paper, the HAM has been successfully applied to obtain the approximate solutions for solving fractional ordinary and partial differential equations. The solutions are obtained in the form of infinite power series which can be written in a closed form with the help of the nonzero convergence control parameter $\hbar$. In view of the results, it is easy to see that HAM is powerful mathematical tool for solving different kinds of linear and nonlinear fractional differential equations arising in different fields of science and engineering. This technique may also be applied for solving systems of non-linear fractional partial differential equations and non-linear fractional integral equations.

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