Limit Cycle Problem for Quadratic System with some Applications of a Class I

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Abstract: This paper is part of a wider study limit cycle problems of a class and planar system; we study the existence of limit cycle for polynomial planar system. In this paper, we present a proof of a result on the existence of limit cycle of the Quadratic System:

$$\frac{dx}{dt} = P_2(x, y), \frac{dy}{dt} = Q_2(x, y)$$

Key words: Quadratic system, Limit cycle, Quadratic polynomials.

Introduction:

We are interested in existence of limit cycle for some polynomial planar system. We discuss the existence by applying theorem1 in [2]. We introduce the Quadratic system and give a Theorem1 for existence of limit cycles. We apply Theorem1 for existence of limit cycles for some applications and verify the exactions.

1. The Existence of Limit Cycles:

We consider the quadratic differential system

$$\frac{dx}{dt} = P_2(x, y), \frac{dy}{dt} = Q_2(x, y) \quad (1)$$

Where P_2 and Q_2 are general quadratic polynomials.

In this case the system (1) under this transformation becomes

$$\frac{dx^l}{dt} = P_2^l(x^l, y^l), \frac{dy^l}{dt} = x^l(ax^l + by^l + c) \quad (2)$$

The system (2) has three classes

$$\text{I. } a = b = 0, c \neq 0,$$

$$\frac{dx}{dt} = k + \delta x + ey + tx^2 + mxy + ny^2, \frac{dy}{dt} = cx \quad (3)$$

$$\text{II. } a \neq 0, b = 0, c \neq 0,$$

$$\frac{dx}{dt} = k + \delta x + ey + tx^2 + mxy + ny^2,$$

$$\frac{dy}{dt} = x(ax + c). \quad (4)$$

$$\text{III. } b \neq 0,$$

$$\frac{dx}{dt} = k + \delta x + ey + tx^2 + mxy + ny^2,$$

$$\frac{dy}{dt} = x(ax + by + c). \quad (5)$$

To change (3- 5) into

$$\text{I. } \frac{dx}{dt} = -y + \delta x + tx^2 + mxy + ny^2, \quad \frac{dy}{dt} = x$$

$$\text{II. } \frac{dx}{dt} = -y + \delta x + tx^2 + mxy + ny^2,$$

$$\frac{dy}{dt} = x(1 + ax), a \neq 0.$$

$$\text{III. } \frac{dx}{dt} = -y + \delta x + tx^2 + mxy + ny^2,$$

$$\frac{dy}{dt} = x(1 + ax + by), b \neq 0.$$

First studied necessary and sufficient conditions for the quadratic system of the form

$$\begin{aligned} \dot{x} &= -y + a_{20}x^2 + a_{11}xy + a_{02}y^2, \\ \dot{y} &= x + b_{20}x^2 + b_{11}xy + b_{02}y^2 \end{aligned} \quad (6)$$

With (0, 0) as its k th-order fine focus (k=1, 2, 3) and then derived necessary and sufficient conditions for (0, 0) to be a center.

Theorem1: For system (6) introduces the quantities

$$\begin{aligned}
W_1 &= A\alpha - B\beta, \\
W_2 &= [\beta(5A - \beta) + \alpha(5B - \alpha)]\gamma, \\
W_3 &= (A\beta + B\alpha)\gamma\delta.
\end{aligned} \tag{7}$$

Where

$$\begin{aligned}
A &= a_{20} + a_{02}, & B &= b_{20} + b_{02}, \\
\alpha &= a_{11} + 2b_{02}, & \beta &= b_{11} + 2a_{20}, \\
\delta &= a_{02}^2 + b_{20}^2 + a_{02}A + b_{20}B, \\
\gamma &= b_{20}A^3 - (a_{20} - b_{11})A^2B + (b_{02} - a_{11})AB^2 - a_{02}B^3.
\end{aligned} \tag{8}$$

Then the following assertions are true:

(a) $(0, 0)$ is a k th – order fine focus ($k=1, 2, 3$) if and only if the following k th group of conditions holds:

$$\begin{aligned}
& \text{(i)} \quad W_1 \neq 0; & \text{(ii)} \quad W_1 = 0, W_2 \neq 0; & \text{(iii)} \quad W_1 = W_2 = 0, W_3 \neq 0 \\
& \tag{9}
\end{aligned}$$

(b) The stability of the k th – order fine focus decided by the sign of W_k : it is stable for $W_k < 0$ and unstable for $W_k > 0$.

(c) The origin is a center if and only if $W_1 = W_2 = W_3 = 0$.

In order to study the stability of $(0, 0)$, when only $W_1 = 0$ we have to examine W_2 , and when only $W_1 = W_2 = 0$ we have to examine W_3 . Hence if we use the quantities

$$\begin{aligned}
W_1 &= A\alpha - B\beta, \\
W_{2'} &= \begin{cases} (5A - \beta)\beta\gamma, & \text{when } A \neq 0, \\ (5B - \alpha)\alpha\gamma, & \text{when } B \neq 0, \\ 0, & \text{when } A = B = 0; \end{cases} \\
W_{3'} &= \begin{cases} A\beta\gamma\delta, & \text{when } A \neq 0 \\ B\alpha\gamma\delta, & \text{when } B \neq 0 \\ 0, & \text{when } A = B = 0 \end{cases}.
\end{aligned} \tag{10}$$

To replace $W_1 = W_2$, and W_3 in (7), then the conclusion of theorem (1) remains unchanged.

Corollary. System (6) takes the origin as its center if and only if at least one of the following four groups of conditions holds:

- 1) $A = B = 0$ (equivalent to $\lambda_1 = \lambda_3 - \lambda_6 = 0$).
- 2) $\alpha = \beta = 0$ (equivalent to $\lambda_1 = \lambda_4 = \lambda_5 = 0$).
- 3) $A\alpha - B\beta = \gamma = 0$ (equivalent to $\lambda_1 = \lambda_2 = \lambda_5 = 0$).
- 4) $5A - \beta = 5B - \alpha = \delta = 0$ (equivalent to $\lambda_1 = \lambda_5 = \lambda_4 + 5\lambda_3 - 5\lambda_6 = \lambda_3\lambda_6 - 2\lambda_6^2 - \lambda_2^2 = 0$).

2. Applications

2.1. We considered the following Quadratic system

$$\begin{aligned}\dot{x} &= -y + (ax + bx^2) \\ \dot{y} &= -(x + bx^2 + cx^3)\end{aligned}\quad (11)$$

For the quadratic system of the form

$$\begin{aligned}\dot{x} &= -y + a_{10}x + a_{20}x^2 + a_{01}y + a_{02}y^2 + a_{11}xy \\ \dot{y} &= x + b_{10}x + b_{20}x^2 + b_{01}y + b_{02}y^2 + b_{30}x^3 + b_{11}xy\end{aligned}\quad (12)$$

For system (11): $a_{10}=a$, $a_{20} = b$, $a_{01} = -1$, $a_{02} = 0$, $a_{11} = 0$

$$b_{01} = b_{02} = 0 , \quad b_{10} = -1 , \quad b_{20} = -b , \quad b_{30} = -c , \quad b_{11} = 0$$

And apply the system (8):

$$A = a_{20} + a_{02} = b + 0 = b ,$$

$$B = b_{20} + b_{02} = -b + 0 = -b ,$$

$$\alpha = a_{11} + 2b_{02} = 0 + 2(0) = 0 ,$$

$$\beta = b_{11} + 2a_{20} = 0 + 2(b) = 2b ,$$

$$\delta = a_{02}^2 + b_{20}^2 + a_{02}A + b_{20}B = b^2 + b^2 = 2b^2 ,$$

$$\begin{aligned}\gamma &= b_{20}A^3 - (a_{20} - b_{11})A^2B + (b_{02} - a_{11})AB^2 - a_{02}B^3 \\ &= (-b)(b)^3 - (b)(-b)(b)^2 = b^4 - b^4 = 0 .\end{aligned}$$

In traduce the quantities. For system (7):

$$w_1 = b(0) - (-b)(2b) = 2b^2$$

$$w_2 = [0(5b) + (0)(-5b)].0 = 0$$

$$w_2 = 0$$

$$w_3 = (b(0) + (-b)(0)).0 = 0$$

$$w_1 \neq 0 , \quad w_2 = w_3 = 0$$

Which is the order fine focus.

2.2. Now we consider the following Quadratic system

$$\dot{x} = k + \delta x + ey + lx^2 + mxy + ny^2$$

$$\dot{y} = x(c + ax + by) \quad (13)$$

Let $k = 0$, $e = -1$, $c = 1$

$$\dot{x} = -y + \delta x + lx^2 + mxy + ny^2$$

$$\dot{y} = x + ax^2 + bxy \quad (14)$$

When $\delta = 0$, the system (14) becomes

$$\dot{x} = -y + \delta x + lx^2 + mxy$$

$$\dot{y} = x + ax^2 + bxy \quad (15)$$

The case: $a = b = 0$

$$\dot{x} = -y + \delta x + lx^2 + mxy$$

$$\dot{y} = x \quad (16)$$

Apply the system (12)

For system (16): $a_{10} = \delta$, $a_{20} = l$, $a_{01} = -1$, $a_{02} = 0$, $a_{11} = m$

$$b_{10} = 1, b_{11} = b_{20} = b_{01} = b_{02} = 0,$$

And apply the system (8)

$$A = a_{20} + a_{02} = l + 0 = l,$$

$$B = b_{20} + b_{02} = 0,$$

$$\alpha = a_{11} + 2b_{02} = m,$$

$$\beta = b_{11} + 2a_{20} = 2l,$$

$$\delta = a_{02}^2 + b_{20}^2 + a_{02}A + b_{20}B = 0,$$

$$\gamma = b_{20}A^3 - (a_{20} - b_{11})A^2B + (b_{02} - a_{11})AB^2 - a_{02}B^3$$

$$\gamma = 0.$$

In traduce the quantities. For system (7):

$$w_1 = lm - 0 = lm$$

$$w_2 = [(2l)(5l - 2l) + m(-m)].0 = 0$$

$$w_2 = 0$$

$$w_3 = (l(2l) + 0(m)).0 = 0.$$

$$w_1 \neq 0, \quad w_2 = w_3 = 0$$

Which is the order fine focus.

When $\delta = 0$, the system (14) becomes

$$\begin{aligned}\dot{x} &= -y + \delta x + lx^2 + ny^2 \\ \dot{y} &= x + ax^2 + bxy\end{aligned}\quad (17)$$

The case: $a = b = 0$

$$\begin{aligned}\dot{x} &= -y + \delta x + lx^2 + ny^2 \\ \dot{y} &= x\end{aligned}\quad (18)$$

Apply the system (12)

For system (18): $a_{10} = \delta$, $a_{20} = l$, $a_{01} = -1$, $a_{02} = n$, $a_{11} = 0$

$$b_{10} = 1, b_{11} = b_{20} = b_{01} = b_{02} = 0,$$

And apply the system (8)

$$A = a_{20} + a_{02} = l + n,$$

$$B = b_{20} + b_{02} = 0,$$

$$\alpha = a_{11} + 2b_{02} = 0,$$

$$\beta = b_{11} + 2a_{20} = 2l,$$

$$\delta = a_{02}^2 + b_{20}^2 + a_{02}A + b_{20}B = n^2 + n(l + n) = 2n^2 + nl,$$

$$\gamma = b_{20}A^3 - (a_{20} - b_{11})A^2B + (b_{02} - a_{11})AB^2 - a_{02}B^3$$

$$\gamma = 0.$$

In traduce the quantities. For system (7):

$$w_1 = (l + n).0 - 0(2l) = 0$$

$$w_2 = [2l(5(l + n) - 2l) + 0].0 = 0$$

$$w_2 = 0$$

$$w_3 = ((l + n)2l + 0).0 = 0$$

$$w_1 = w_2 = w_3 = 0$$

Which is the origin is a center.

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