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On Contact Lorentzian and Contact Pseudo-Riemannian Manifolds

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Abstract: we introduce a systematic study contact Lorentzian Pseudo structures with Lorentzian Pseudo-metric manifolds and emphasizing analogies and differences with respect to the Lorentzian case. We investigate Pseudo- sphere and the Pseudo-hyperbolic space are the only simply connected Sasakian Lorentzian Pseudo-metric manifolds of constant sectional curvature and we classify contact Lorentzian pseudo-metric manifolds of constant sectional curvature of three-dimensional.

Keywords: Pseudo- sphere, Pseudo-hyperbolic space, manifolds, three-dimensional

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1. Introduction

The contact metric structures are a well-known intensively studied research field in differential geometry. Were defined by Blair [D.E. Blair(2002)], gives a wide thought and provides detailed informations in this area and introduced some results obtained in this framework. Contact Lorentzian and contact Pseudo-metric structures (η, g), where η is a contact 1-form and g are a metric associated with it, are a generalization of contact metric structure [G. Calvaruso(2011), G. Calvaruso, and D. Perrone(2010), T. Takahashi(1969), Mohamed H. A. Hamed, Islam F. M. Osman, Mohammed B.A. Mohammad, Arafat Abdelhameed Abdelrahmann and Asmaa Eltayeb Ali Elhassan(2021)]. Contact structure with Pseudo-Riemannian metrics were introduced and studied in [T. Takahashi(1969)], with a focus on the Sasakian case see G. Calvaruso and D. Perrone in [G. Calvaruso, and D. Perrone(2010)].

In this paper, we focus on the relevant cases of contact Lorentzian and contact Pseudo-Riemannian structures are called contact Lorentzian Pseudo-Riemannian manifolds. We introduce the relationship between contact Lorentzian and contact Pseudo-Riemannian manifolds. We prove some general results and introduce several explicit examples for both cases. The rest of the paper is organized as the follows.

In Section 2, we will present the basic notation of the contact Lorentzian manifolds and contact Pseudo-metric manifolds. We introduce deformations of contact Lorentzian and Pseudo-metric structures it that in Section 3. In Section 4, we will study and clarification the contact Lorentzian Pseudo-Riemannian manifolds of contact sectional curvature, and given some examples.

2. PRELIMINARIES

An almost contact (ϕ, ξ, η) on odd-dimensional smooth manifold M, where ϕ is a

$(1, 1) -$ tensor field, ξ is a global vector field and η a global differential 1-form, such that

$\eta V d\eta = 0,$, and satisfying

$\eta(\xi) = 0, \phi 2 = -I + \eta \otimes \xi, \phi(\xi) = 0, \eta \circ \phi = 0$ (2.1)

and $Rank(\phi) = 2n$.

Let now g denote a Riemannian metric and associated of M, then, g is said to be compatible

with the almost contact structure (ϕ, η, ξ) [S. Dragomir, M. Hasan and F. Al-Solamy(2016)] if

$g(\phi X, \phi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y),$ (2.2)

where $\epsilon = \pm 1$, a smooth manifold M, equipped with an almost contact structure (ϕ, η, ξ) and compatible Pseudo-Riemannian metric g will be called an almost contact Pseudo metric manifold [G. Calvaruso, and D. Perrone(2010)].

Then from (2.1) and (2.2), we obtain $\eta(X) = \epsilon g(\xi, X)$. In particular, if $g(\xi, \xi) = \epsilon$, the characteristic vector field ξ is either spacelike (if) $\epsilon = 1$ or time-like (if) $\epsilon = -1$, but ξ is never lightlike(null), furthermore from (2.2) implies that

$g(\phi X, Y) = -g(X, \phi Y)$ [G. Calvaruso, and D. Perrone(2010)], but if M equipped with an almost contact structure (ϕ, η, ξ) and compatible Lorentzian metric g will be called an almost contact Lorentzian manifold [G. Calvaruso(2011)]. Since

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y) \tag{2.3}$$

Note that, by (2.1) and (2.3), $\eta(X) = -g(\xi, X)$, in particular, $g(\xi, \xi) = -1$, then, ξ is timelike [G. Calvaruso(2011)].

Let M^{2n+1} be an almost contact Lorentzian Pseudo-metric manifolds endowed with almost

contact structure (ϕ, ξ, η) and compatible Lorentzian Pseudo-Riemannian g [D.E. Blair(2002)].

We consider now $M^{2n+1} \times \mathbb{R}$ and, we denote a vector field $M^{2n+1} \times \mathbb{R}$ by $(X, f\ \frac{d}{dt})$

Where X is tangent to M^{2n+1}, t the coordinate on $\mathbb{R}$, and f a C^{∞} function on $M^{2n+1} \times \mathbb{R}$.

Define almost complex structure (ϕ, ξ, η) by $J\left(X, f\ \frac{d}{dt}\right) = \phi X - f\xi, \eta(X)\frac{d}{dt}$

if J is integrable the almost contact structure (ϕ, ξ, η) is called normal [G. Calvaruso(2011)].

since the vanishing of the Nijenhuis torsion of J is a necessary and sufficient condition of integrability,

$$[J, J]\big((X, 0), (Y, 0)\big)$$

$$= -([X, Y], 0) + \left[\left(\phi X, \eta(X)\frac{d}{dt}\right), \left(\phi Y, \eta(Y)\frac{d}{dt}\right)\right]$$

$$- J\left[\left(\phi X, \eta(X)\frac{d}{dt}\right), (Y, 0)\right] - J\left[(X, 0), \left(\phi Y, \eta(Y)\frac{d}{dt}\right)\right]$$

$$= \Big([\phi, \phi], (X, Y) + 2d\eta(X, Y)\xi, \big((£\phi X\eta)(Y)\big)$$

$$- (£\phi X\eta)(X)\frac{d}{dt}\Big), [J, J]\left((X, 0), (0, \frac{d}{dt}\right)$$

$$= \left[\left(\phi X, \eta(X)\frac{d}{dt}\right), (-\xi, 0)\right] - J\left[\left(\phi X, \eta(X)\frac{d}{dt}\right), (0, \frac{d}{dt}\right]$$

$$- J[(X, 0), (-\xi, 0)]$$

$$= \left[-[\phi X, \xi], \big(\xi\ \eta(X)\big)\frac{d}{dt}\right] + \left(\phi[X, Y], \eta([X, Y])\frac{d}{dt}\right)$$

$$= ((£\xi\phi)X, (£\xi\eta)(X)),$$

where $[J, J]$ is tensor field of type (1, 2) and $£$ is a Lie derivative.

Now we define four tensors respect to ϕ by

$$N^{(1)} = [\phi,\phi] + 2d\eta\xi, \qquad N^{(2)} = (\pounds\phi X\eta)(Y) - (\pounds\phi Y\eta)(X),$$
$$N^{(3)} = (\pounds\xi\phi)X, \quad N^{(4)} = (\pounds\xi\eta)X.$$

Furthermore, the vanishing of $N^{(1)}$ implies the vanishing of $N^{(2)}$, $N^{(3)}$

and $N^{(4)}$ [S. Sasaki, Y. Hatakeyama(1962)]. Then $[\phi,\phi] + 2d\eta\xi = 0$ is a necessary and sufficient condition for the integrability of J.

Lemma 2.1. For an almost contact metric structure hence, g a compatible

Pseudo-metric on M^{2n+1}. Then,

$$2g((\nabla X\phi)Y,Z) = 3d\Phi(X,\phi Y,\phi Z) - 3d\Phi(X,Y,Z) + g(N^{(1)}(Y,Z),\phi X) + N^{(2)}(Y,Z)\eta(X) + 2d\eta(\phi Y,X)\eta(Z) - 2d\eta(\phi Z,X)\eta(Y),$$

where X, Y and Z are tangent vector fields, Φ is a 2-form, then, we can put

$$\Phi(X,Y) = g(X,\phi Y).$$

Proof.

Recall that the Koszul formula of a metric g is given by

$$2g((\nabla X\phi)Y,Z) = Xg(Y,Z) + Y g(X,Z) - Zg(X,Y) + g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y).$$

Then, from (2.2) we get, $g(X,Y) = \Phi(\phi X,Y) - \eta(X)\eta(Y)$.

By a direct calculation, which completes the proof. If the compatible Lorentzian Pseudo metric g satisfies

$$g(X,\phi Y) = (d\eta)(X,Y), \qquad (2.4)$$

where η is a contact on M, ξ the associated vector field and g is an associated metric, then

the structure (M,ϕ,ξ,η,g) is called contact Lorentzian Pseudo-Riemannain manifolds.

Assume that (M^{2n+1},η) is a contact Lorentzian Pseudo-metric manifolds hence, there exist the Levi-Civita connection on M denoted by ∇.

Then from (2.1) and (2.4), we have $(d\eta)(\xi,X) - g(X,\phi\xi) = 0$ and

$N^4 = \pounds\eta = d\circ i_\xi\eta + i_\xi d\eta = d\eta(\xi) + d\eta(\xi,.) = 0$. Since N^4 is vanishing, then, from

$\pounds_\xi \eta = 0$, we get $0 = (\pounds_\xi \eta)X = \xi g(\xi, X) - g(\xi, [\xi, X]) = g(\nabla \xi_\xi, X)$

for any vector field X. Thus, $\nabla \xi_\xi = 0$, then the integral curves of a vector field ξ are

geodesic. Furthermore, then, from (2.4) we get $d\Phi = 0$.

Corollary 2.2. [D.E. Blair(2002)**]** In a contact metric structure the formula of Lemma 2.1 be in the formula

$$2g(\nabla_X \phi)Y, Z) = g(N^1(Y, Z), \phi X) + 2d\eta(\phi Y, X)\eta(Z) - 2d\eta(\phi Z, X)\eta(Y). \quad (2.5)$$

Note that on contact Lorentzian Pseudo-metric manifolds, $N^{(3)} = 0$ if and only if ξ is a killing vector field [G. Calvaruso(2011)]. Then from (2.4) and $f_\xi \eta = 0$, we obtain

$(\pounds_\xi d\eta)(X, Y) = (\pounds_\xi g)(X, \phi Y) + g(X, (\pounds_\xi \phi)Y) = 0$, and then,

$\pounds_\xi g = 0 \Longleftrightarrow \pounds_\xi \phi = 0$. Then we can use the tensor

$$h = \pounds_\xi \phi = N^{(3)} \quad (2.6)$$

Now, using (2.5) and direct calculation, we can derive properties of the covariant derivative

$$\nabla_\xi \phi = 0, \quad (2.7)$$

$$-\nabla_X \xi = \epsilon \phi X + \phi h X. \quad (2.8)$$

Exactly similar to the Riemannian case, from (2.7) and (2.8), we get

$h\phi = -h\phi$ and $h\xi = trh = 0$. Further, assume that $\tau = \pounds_\xi g$, then, τ and h related by

$$\tau(X, Y) = 2g(X, h\phi Y) = 2g(h\phi X, Y).$$

Lemma 2.3. [G. Calvaruso, and D. Perrone(2010)**]** In a contact Lorentzian Pseudo-metric manifold (M^{2n+1}, η, g),

$$div\, \xi = 0, div\, \eta = 0.$$

Proof. Suppose that a local ϕ-basis $[\xi, E_1, \dots, E_{2n}] = [\xi, e_1, \dots, e_n, \phi e_1, \dots, \phi e_n]$ on the contact Lorentzian Pseudo-Riemannian manifolds M^{2n+1}. So $\nabla_\xi \xi = 0$ and $h\phi = -h\phi$,

from (2.1) and (2.8) we obtain

$$div\,\xi = tr\nabla\xi = \sum_{i=1}^{n} g(\nabla e_i\xi, e_i) + \sum_{i=1}^{n} g(\nabla\phi e_i\xi, \phi e_i)$$
$$= -\sum_{i=1}^{n} g(\phi h e_i\xi, e_i) - \sum_{i=1}^{n} g(\phi h\phi e_i\xi, \phi e_i) = 0$$

Furthermore, we get $div\,\eta = -tr\nabla\eta = -\epsilon\, div\,\xi = 0.$

Definition 2.4. A contact Lorentzian Pseudo-Riemannian manifolds (M, η, g) is called Sasakian if it is normal , (that's it $[\phi, \phi] + 2d\eta \otimes \xi = 0$), and if $h = 0$ is called K −contact, (ξ is a Killing vector field).

Corollary 2.5. A Sasakian Lorentzian Pseudo-Riemannian manifold is K −contact, refer to [G. Calvaruso(2011), G. Calvaruso, and D. Perrone(2010)].

Theorem 2.6. [G. Calvaruso, and D. Perrone(2010)**]** A contact Lorentzian Pseudo-Riemannian manifolds M^{2n+1} is K-contact if satisfies the condition $\varrho(\xi, \xi) = 2n$.

In a Riemannain case and Pseudo-Riemannian, we note that $trh^2 = 0$, implies $\varrho(\xi, \xi) = 2n$.

The K −contact Riemannian manifolds characterized by $\varrho(\xi, \xi) = 2n$, implies that $trh^2 = 0$ and so, $h = 0$ if it was h is diagonalizable [D.E. Blair(1977)].

3. Deformations of Contact Lorentzian And Contact Pseudo-Metric Structures

Suppose that (M^{2n+1}, η, g) be a contact Lorentzian Pseudo-metric manifolds, consider the deformed structures

$$\bar{\eta} = a\eta, \quad \bar{\xi} = \frac{1}{a}\xi, \quad \bar{\phi} = \phi, \quad \bar{g} = ag + a(a-1)\eta \otimes \eta, \qquad (3.1)$$

where a is a real constant $a > 0$. Characterization other contact Pseudo-metric structure on M^{2n+1}, having the contact distribution $ker\,\bar{\eta} = ker\,\eta$, which a deformation is called a D-homothetic deformation, then from (3.1) is the Lorentzian Pseudo-Riemannain counterpart of D-homothetic deformation of a contact Riemannain structure [S. Tanno(1968)], where $g(\xi, \xi) = \epsilon$, from (3.1), we can note that,

$\bar{g}(\bar{\xi}, X) = \epsilon\bar{\eta}(X)$. Now, let $\bar{\nabla}$ is Levi-Civita connection of $\bar{g}$ from (3.1) the Lorentzian Pseudo-Riemannian metric

$$\tilde{g} = \frac{1}{a}\bar{g} = g - (a-1)\eta \otimes \eta \qquad (2.3)$$

where $\bar{g}$is Riemannian and a positive constant $a > 0$ [S. Tanno(1968)], the Lorentzian Pseudo-metrics homothetic to $\bar{g}$. So, $\bar{\nabla} = \tilde{\nabla}$ and $\tilde{R} = R$ by (3.2) and the Koszul formula, we get

$$\tilde{g}\left(\tilde{\nabla}_X Y, Z\right) = g(\nabla_X Y, Z) - (a-1)\eta(Z)X(\eta(Y)) - \eta(Z)g(X,\phi Y) - \eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z),$$

from (3.2) and (2.8), we have

$$\bar{\nabla}_X Y = \nabla_X Y - \frac{a-1}{a}\eta(\nabla_X Y)\xi + \frac{a-1}{a}\left[X\left(\eta(Y)\right) - g(X,\phi Y)\right] - \epsilon(\mathrm{a}-1)[\eta(X)\phi Y + \eta(Y)\phi X] \qquad (3.3)$$

Then, by using $\eta(X) = \epsilon g(X,\xi)$ and (2.8), therefore, the equation (3.3) becomes

$$\bar{\nabla}_X Y = \nabla_X Y + \epsilon\frac{a-1}{a}g(hX,\phi Y)\xi - \epsilon(\mathrm{a}-1)[\eta(X)\phi Y + \eta(Y)\phi X] \qquad (3.4)$$

Theorem 3.1. A D-homothetic deformation of a K-contact Lorentzian and Pseudo-metric structures is also a K −contact

Proof. By using (3.1), $\bar{h} = \frac{1}{2}\pounds_{\bar{\xi}}\bar{\phi} = \frac{1}{2a}\pounds_{\xi}\phi = \frac{1}{a}h$ this,), $\bar{h} = 0$ if and only if $h = 0$, since (η, g) is K −contact if and only if (η, g) is K −contact.we can to refer [G. Calvaruso(2011), G. Calvaruso, and D. Perrone(2010)], for more information of K −contact of Lorentzian and Pseudo contacts

Proposition 3.2. Let (M,ϕ,ξ,η,g) be a contact Lorentzian Pseudo-metric manifolds and (ϕ,ξ,η,g) the D-homothetic deformation as in the equation (3.1). then,

$$\bar{\ell}X = \frac{1}{a^2}[\ell\mathrm{x} + (a^2-1)\phi^2 X - 2\epsilon(\mathrm{a}-1)hX], \qquad (3.5)$$

$$\begin{aligned}\bar{R}(X,Y)Z = R(X,Y)Z &- \frac{a-1}{a}\left[\eta(R(X,Y)Z) + g(X,(\nabla_Y\phi)Z) - g(Y,(\nabla_X\phi)Z)\right)\xi \\ &+ \frac{a-1}{a}[(X,\phi Z)(\epsilon a\phi Y + \phi hY) - g(Y,\phi Z)(\epsilon a\phi X + \phi hX)] \\ &- \epsilon(\mathrm{a}-1)\eta[X,Y]\phi Z \\ &+ \frac{a-1}{a}[\eta(\nabla_X Z)\phi hY - \eta(\nabla_Y Z)\phi hX] \qquad (3.6)\end{aligned}$$

where $\forall X,Y,Z \in ker\,\eta = ker\,\overline{\eta}$.

Proof. By using (3.4) to calculate $\bar{\nabla}_{\bar{\xi}}\bar{h}$ and $\ell := R(.,\xi)\xi = -\phi(\nabla_\xi h) + \phi^2 + h^2$ to express $\nabla_\xi h$, in terms of ℓ. Hence (3.5) follows from (3.3) by calculation, then, from (3.1) we get $\bar{g}(X,Y) = ag(X,Y),\ \forall X,Y \in ker\,\eta$

Definition 3.3. Let (M,ϕ,ξ,η,g) be a contact Lorentzian Pseudo-metric manifolds. We can put

$$k(\xi,X) = \frac{R(X,\xi,X,\xi)}{\epsilon g(X,X)} = -\frac{g(\ell X,X)}{\epsilon g(X,X)} \quad and \quad k(X,\phi X) = \frac{R(X,\phi X,X,\phi X)}{g(X,X)^2}$$

where $X \in ker\,\eta,\ k(\xi,X)\ and\ k(X,\phi X)$ are called the ξ −sectional curvature and the ϕ −sectional curvature respectively, its determined by X.

4. Contact Lorentzian Pseudo-Metric Manifolds of Contact Sectional Curvature

The main aim of this section is to study the contact Lorentzian and contact Pseudo metric manifolds, and to prove the following fundamental results, for more details refer [T. Takahashi(1969), Z. Olszak(1979)].

Theorem 4.1. Let (M^{2n+1},η,g) be a contact Lorentzian Pseudo-metric manifold, such that $n \geqslant 1$. If (M^{2n+1},g) is of constant sectional curvature K, then (M^{2n+1},η,g) is Sasakian.

Theorem 4.2. Let (M^{2n+1},η,g) be a contact Lorentzian Pseudo-metric manifolds, then,

$$(\nabla_X\phi)Y + (\nabla_{\phi X}\phi)\phi Y = 2g(X,Y)\xi - \eta(Y)\{\epsilon X + \epsilon\eta(X)\xi + hX\}. \qquad (4.1)$$

Proof. If $\tilde{X},\tilde{Y} \in ker\,\eta$, using (2.5) we get

$$\begin{aligned} 2g\left((\nabla_{\tilde{X}}\phi\tilde{Y}),Z\right) &= -g\big([\tilde{Y},Z],\phi\tilde{X}\big) + g\big([\phi\tilde{Y},\phi Z],\phi\tilde{X}\big) - g\big([\phi\tilde{Y},Z],\tilde{X}\big) \\ &\quad - g\big([\tilde{Y},\phi Z],\tilde{X}\big) + 2\epsilon g\big(\tilde{X},\tilde{Y}\big)\eta(Z), \end{aligned}$$

and, by similar a way,

$$\begin{aligned} 2g\left((\nabla_{\phi\tilde{X}}\phi\tilde{Y}),Z\right) &= -g\big([\phi\tilde{Y},Z],\phi^2\tilde{X}\big) + g\big([\phi^2\tilde{Y},\phi Z],\phi^2\tilde{X}\big) - g\big([\phi^2\tilde{Y},Z],\phi\tilde{X}\big) \\ &\quad - g\big([\phi\tilde{Y},\phi Z],\phi\tilde{X}\big) + 2\epsilon g\big(\phi\tilde{X},\phi\tilde{Y}\big)\eta(Z). \end{aligned}$$

From the two previous equations, if Z is arbitrary, we obtain

$$\left(\nabla_{\tilde{X}}\phi)\tilde{Y}\right) + \left(\nabla_{\phi\tilde{X}}\phi)\phi\tilde{Y}\right) = 2g\left(\tilde{X},\tilde{Y}\right)\xi \qquad (4.2)$$

for any vector fields $\tilde{X},\tilde{Y} \in ker\,\eta$. If X,Y are arbitrary tangent vector fields, then, $X = \tilde{X} + \eta(X)\xi$ and $Y = \tilde{Y} + \eta(Y)\xi$, then, we find,

$$(\nabla_X\phi)Y = \left(\nabla_{\tilde{X}}\phi)\tilde{Y}\right) + \eta(Y)\{-\epsilon X + \epsilon\eta(X)\xi - hX\},$$

which implies $(\nabla_{\phi X}\phi)\phi Y = (\nabla_{\phi\tilde{X}}\phi)\phi\tilde{Y}$.

Then from the last two equations and using (4.2), we get (4.1) which complete the proof.

Example 4.3. [G. Calvaruso(2011)**]** Let $({R_{\{2\}}}^{2n+1} \equiv {C_{\{1\}}}^{n+1}, \bar{g}, J)$ be a Pseudo-Euclidean space with an indefinite Ka¨hler structure, we suppose the Pseudo-hyperbolic space

$${H_{\{1\}}}^{2n+1}(-1) = [\ \{\chi \in {R_{\{2\}}}^{2n+1} : \bar{g}(\chi,\chi) = -1],$$

which represents a hyperquadric of ${R_{\{2\}}}^{2n+1}$ of odd dimensional-$(2n + 1)$, index 1 and constant sectional curvature -1. There exists a canonical structure $(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ of Sasaian Lorentzian Pseudo manifold, given by the following tensors formula:

$\bar{g} = \tilde{g}|{H_{\{2s-1\}}}^{2n+1}(-1)$, $\bar{\xi}: \chi \in {H_{\{2s-1\}}}^{2n+1}(-1) - J\chi$, $\bar{\eta}(X) = -g(\bar{\eta}, X)$, $\bar{\phi} = \bar{\pi} \circ J$, where $\bar{\pi}(X) = X + \tilde{g}(X,\chi)\chi$ and ${H_{\{1\}}}^{2n+1}$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^{2n}$ and as well is not connected. Note that, in [Z. Olszak(1979)] obtained a result in Riemannain frame, proving that in any dimensional ≥ 5, if (M^{2n+1}, η, g) is a contact Riemannain manifold has constant sectional curvature K, then $K = 1$ and (M^{2n+1}, η, g) is Sasakian. The previous result was extended to Pseudo-Riemannian frame in [G. Calvaruso, and D. Perrone(2010)], also was extended in the Lorentzian frame in [G. Calvaruso(2011)].

Corollary 4.4. For any $n \geq 2$, the Pseudo- sphere ${\mathbb{S}_{\{2s\}}}^{2n+1}(1)$ and the Pseudo-hyperbolic space ${H_{\{2s-1\}}}^{2n+1}(-1)$ are the only simply connected Sasakian Lorentzian Pseudo-metric manifolds of constant sectional curvature.

Now we present the classification of three-dimensional contact Lorentzian Pseudo-Riemannian manifolds of constant sectional curvature. In this case, we will study the three-dimensional homogeneous contact Lorentzian manifolds of constant sectional curvature, we can clarify that by the following example.

Example 4.5. On $\mathbb{R}^3(X, Y, Z)$, we consider the Lorentzian metric defined by

$$g = \frac{1}{4}dX^2 - dY \otimes\ dZ, \quad (g_{ij}) = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{pmatrix}. \tag{4.3}$$

The 1 $-$form $\eta = \frac{1}{2}(e^X dY + e^{-X} dZ)$.

Now, we get

$$d\eta = \frac{1}{4}\begin{pmatrix} 0 & e^X & -e^{-X} \\ -e^X & 0 & 0 \\ e^{-X} & 0 & 0 \end{pmatrix}, \tag{4.4}$$

such that $\eta \wedge (d\eta) \neq 0$. Thus, η is a contact form, the vector field ξ, satisfies the following conditions:

$\eta(\xi) = 1$ and $(d\eta)(\xi, .) = 0$, and ξ given by $\xi = e^{-X}\partial Y + e^X \partial Z$.

Then, from (5.4) implies that $g(\xi, \xi) = -1$, the contact distribution is spanned by vector fields $E_1 = e^{-X}\partial Y - e^X \partial Z$, $E_2 = 2\partial X$ and $[\xi, E_1, E_2]$ is a Pseudo-orthonormal basis

for g. if there exist tensor field ϕ, of type $(1,1)$ with respect to the basis $\{\partial X, \partial Y, \partial Z\}$, defined by

$$\phi = \begin{pmatrix} 0 & e^X & -e^{-X} \\ e^{-\frac{X}{2}} & 0 & 0 \\ e^{\frac{X}{2}} & 0 & 0 \end{pmatrix}. \tag{4.5}$$

Now, from above equations, we get $d\eta = g(., \phi)$ and $[\xi, E_1, E_2]$ is a ϕ $-$basis and $E_2 = \phi E_1$. Therefore, (η, ϕ, ξ, g) is a contact Lorentzian structure on $\mathbb{R}^3$. since $[\xi, E_1, E_2]$ is a Pseudo-orthonormal basis given by

$$\{\xi, E_1\} = 0, \quad \{\xi, E_2\} = 2E_1, \quad \{E_1, E_2\} = -2\xi. \tag{4.6}$$

If ∇ is Levi-Civita connection of (M, g), we obtain $\nabla_{E_2}\xi = -2E_1$, $\nabla_{E_2}E_1 = 2\xi$. The curvature tensor satisfies $R(E_1, E_2) = R(E_2, \xi) = R(\xi, E_1)$, then, (η, g) is a flat contact Lorentzian structure on $\mathbb{R}^3$, since the vector field ξ is not Killing vector field, as $\nabla_{E_2}\xi = -2E_1$, from (4.6) we get $hE_1 = E_1$ and $hE_2 = -E_2$.

Theorem 4.6. [G. Calvaruso, and D. Perrone(2010)**]** Let (M, η, g) a three-dimensional locally symmetric contact Lorentzian Pseudo-metric manifolds is either flat or of constant sectional curvature K.

Corollary 4.7. The Pseudo-Euclidean space $\mathbb{R}^3$ and the universal covering of the pseudo-hyperbolic space ${H_1}^3\,(-1)$ are the only three-dimensional connected symmetric contact Lorentzian manifold.

Lemma 4.8. If a contact Lorentzian pseudo-metric manifold is locally symmetric, then, $\nabla_\xi h \;=\; 0$. Note that, if (M, η, g) is locally symmetric, then

$$R(X,\xi)\xi \;=\; -X \;+\; \eta(X)\xi \;+\; h^2X, \qquad (4.7)$$

where X is tangent vector.

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