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Group Classification Invariant Solutions of Burgers' Equation

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Abstract:

The aims of the present paper is to solve the problem of the group classification of the general Burgers' equation $u_t = f(x,u)u_x^2 + g(x,u)u_{xx}$, where f *and* g are arbitrary smooth functions of the variables x *and* u, by using Lie method. The paper is one of the few applications of an algebraic approach to the problem of group classification: We followed the analysis mathematical method using the method of preliminary group classification. A number of new interesting nonlinear invariant models which have nontrivial invariance algebras are obtained. The result of the work is a wide class of equations summarized in table form.

Key words: Infinitesimal generator, Symmetries of Burgers' equation, optimal system, group classification of algebra.

Introduction:

It is well known that the symmetry group method plays an important role in the analysis of differential equations. The history of group classification methods goes back to Sophus Lie. The first paper on this subject is S.Lie ,Arech(1881), where Lie proves that a linear two-dimensional second-order PDE may admit at most a three-parameter invariance group (apart from the trivial infinite parameter symmetry group, which is due to linearity). He computed the maximal invariance group of the one-dimensional heat conductivity equation and utilized this symmetry to construct its explicit solutions. Saying it the modern way, he performed symmetry reduction of the heat equation. Nowadays symmetry reduction is one of the most powerful tools for solving nonlinear partial differential equations (PDEs). Recently, there have been several generalizations of the classical Lie group method for symmetry reductions. L.V.Ovsiannikov (1982), developed the method of partially invariant solutions. This approach is based on the concept of an equivalence group. Which is a Lie transformation group acting in the extended space of independent variables, functions and their derivatives. And preserving the class of partial differential equations. The investigation of the exact solutions plays an important role in the study of nonlinear physical systems. A wealth of methods have been developed to find those exact physically significant solutions of a PDE though it is rather difficult. Some of the most important methods are the inverse scattering method R.M.Miura (1967), Darboux and $Böcklund$ transformations Y.S.Li. (1999), $Hirota$ bilinear method R.Hirota.J.Satsuma(1976) Lie symmetry analysis B.J. Cantwell (2002), etc. the paper H.L.J.Liu.J Zhangb(2008), based on the Lie group method. Is investigated a very famous an important equation, which is the general Burgers' equation as the form

$$u_t = au_x^2 + bu_{xx}, \tag{1.1.1}$$

where $u = u(x,t)$ is the unknown real function. $a, b \in R \; and \; ab \neq 0$. In the present paper. We consider the general Burgers' equation as the form

$$u_t = f(x,u)u_x^2 + g(x,u)u_{xx}. \tag{1.1.2}$$

Where $u = u(x,t)$ is the unknown real function, $f \; and \; g$ are arbitrary smooth functions of the variables $x \; and \; u$. Eq. (1.1.2) represents the Burgers' equation combining both dissipative and nonlinear effects, therefore appears in a wide variety of physical applications. So it is important to lucubrate the exact explicit solutions and similarity reductions for this equation M.Nadjafikhah (2008). Here, we got the preliminary group classification of Eq. (1.1.2) by means of Lie point symmetry, and the constructed optimal systems of subalgebras. The knowledge of the optimal system of subalgebras gives the possibility of constructing the optimal system of solutions M.L.G,M.T.A.Valenti (2004) and permits the generation of new solutions starting form invariant or non-invariant solutions.

(1.1) Symmetries of Burgers' Equations: Let a partial differential equation contains p dependent variables and q independent variables. The one parameter Lie group of transformations $\tilde{x} = x_i + c\xi^i(x,u) + o(c^2)$: $\tilde{u}_\alpha = u_\alpha + \epsilon\varphi^\alpha(x,u) + o(\epsilon^2)$, (1.2.3)

Where $\xi^i = \frac{\partial \tilde{x}_i}{\partial \epsilon}|_{\epsilon=0}, i = 1, \dots, p, and\ \varphi^\alpha = \frac{\partial \tilde{u}_\alpha}{\partial \epsilon}|_{\epsilon=0}, \alpha = 1, \dots, q$, are given.

The action of the Lie group can be recovered from that of its associated infinitesimal generators, we consider general vector field

$$V = \sum_{i=1}^{p} \xi^i\,(x,u)\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \varphi^\alpha\,(x,u)\frac{\partial}{\partial u^\alpha}. \qquad (1.2.4)$$

On the space of independent and dependent variables. Therefore, the characteristic of the vector field V given by (1.2.4) is the function

$$Q^\alpha\big(x,(u)^{(1)}\big) = \varphi^\alpha(x,u) - \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial u^\alpha}{\partial x_i}, \qquad \alpha = 1, \dots, q. \qquad (1.2.5)$$

The second prolongation of infinitesimal operator

$$X = \xi^i(x,t,u)\frac{\partial}{\partial x} + \xi^2(x,t,u)\frac{\partial}{\partial t}\varphi(x,t,u)\frac{\partial}{\partial u}. \qquad (1.2.6)$$

Obtained via the following prolongation formulas:

$$X^{(2)} = X + \varphi^x\frac{\partial}{\partial u_x} + \varphi^t\frac{\partial}{\partial u_t} + \varphi^{xt}\frac{\partial}{\partial u_{xx}}.$$

The coefficients are obtained by

$$\varphi^t = D_t Q + \xi^i u_{xi} + \xi^2 u_{ti}, \quad \varphi^{tJ} = D_t\big(D_J Q\big) + \xi^1 u_{xiJ} + \xi^2 u_{tiJ}. \qquad (1.2.7)$$

Where $Q = \varphi - \xi^1 u_x - \xi^2 u_t$ is the characteristic of the vector field V given by (1.2.4). For instance

$$\varphi^x = D_x\varphi - u_x D_x\xi^1 - u_t D_x\xi^2, \qquad (1.2.8)$$

$$\varphi^t = D_t\varphi - u_x D_t\xi^1 - u_t D_t\xi^2, \qquad (1.2.9)$$

$$\varphi^{xx} = D_x\varphi^x - u_{xx} D_x\xi^1 - u_{xt} D_x\xi^2, \qquad (1.2.10)$$

$$\varphi^{xt} = D_x\varphi^x - u_{xx} D_t\xi^1 - u_{xt} D_t\xi^2, \qquad (1.2.11)$$

where the operators D_x and D_t denotes the total derivatives with respect to x and t:

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + \cdots,$$

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + \cdots,$$

by the theorems, in Y.S.Li. (1999). $X^{(2)}[u_t - f(x,u)u_x^2 - g(x,u)u_{xx}]|_{(1,2)} = 0$. Since

$$X^{(2)}[u_t - f(x,u)u_x^2 - g(x,u)u_{xx}] = \varphi^t - (f_x\xi^1 + f_u\varphi)u_x^2 - (g_x\xi^1 + g_u\varphi)u_{xx} - 2f\varphi^x u_x - g\varphi^{xx},$$

therefore we obtain the following determining function:

$$[\varphi^t - (f_x\xi^1 + f_u\varphi)u_x^2 - (g_x\xi^1 + g_u\varphi)u_{xx} - 2f\varphi^x u_x - g\varphi^{xx}]|_{(1,2)} = 0. \quad (1.2.12)$$

In the case of arbitrary $f(x,u)$ and $g(x,u)$ it follows that

$$\xi^1 = \varphi = \varphi^x = \varphi^t = \varphi^{xx} = 0, \quad (1.2.13)$$

Or, $\xi^1 = \varphi = 0, \qquad \xi^2 = c.$ (1.2.14)

Therefore, for arbitrary $f(x,u)$ and $g(x,u)$ Eq. (1.1.1).admits the one-dimensional Lie algebra g_1, with the basis

$$X_2 = \frac{\partial}{\partial t}. \quad (1.2.15)$$

g_1 is called the principle Lie algebra for Eq. (1.1.1). So, it is remained to specify the coefficients f and g such that Eq. (1.1.1) admits an extension of the principal algebra g_1. Usually, the group classification is obtained by inspecting the determining equation. But in our case the complete solution of the determining equation (1.2.12) is a wasteful venture. Therefore, we don't solve the determining equation but, instead we obtain a partial group classification of Eq. (1.1.1) via the so-called method of preliminary group classification. This method was applied when an equivalence group is generated by a finite-dimensional Lie algebra g_s. The essential part of the method is the classification of all nonsimilar subalgebras of g_s. Actually, the application of the method is simple and effective when the classification is based on finite-dimensional equivalence algebra g_s.

(1.2) Equivalence Transformations: An equivalence transformation is a nondegenerate change of the variables t, x, u taking any equation of the form (1.1.1) into an equation of the same form, generally speaking, with different $f(u,x)$ *and* $g(x,u)$. The set of all equivalence transformations forms an equivalence group s. we shall find a continuous subgroup s_c of it making use of the infinitesimal method. We consider an operator of the group $s_c$in the form

$$Y = \xi^1(x,t,u)\frac{\partial}{\partial x} + \xi^2(x,t,u)\frac{y}{\partial t} + \varphi(x,t,u)\frac{\partial}{\partial u} + \mu(x,t,u,f,g)\frac{\partial}{\partial f} + v(x,t,u,f,g)\frac{\partial}{\partial g}, \quad (1.3.16)$$

from the invariance conditions of Eq. (1.1.1) written the system:

$$u_t - f(x,u)u_x^2 - g(x,u)u_{xx} = 0, \quad (1.3.17)$$

$$f_t = g_t = 0, \quad (1.3.18)$$

where $u, f\ and g$ are considered as differential variables: u on the space (x,t) and f, g on the extended space (x,t,u). The invariance conditions of the system $(1.3.17)$ are

$$Y^{(2)}(u_t - f(x,u)u_x^2 - g(x,u)u_{xx}) = 0. \quad (1.3.19)$$

$$Y^{(2)}(f_t) = Y^{(2)}(g_t) = 0\ ,$$

where $Y^{(2)}$ is the prolongation of the operator (3.16):

$$Y^{(2)} = Y + \varphi^x\frac{\partial}{\partial u_x} + \varphi^t\frac{\partial}{\partial u_t} + \varphi^{xt}\frac{\partial}{\partial u_{xt}}\varphi^{xx}\frac{\partial}{\partial u_{xx}} + \mu^t\frac{\partial}{\partial f_t} + v^t\frac{\partial}{\partial g_t}. \quad (1.3.20)$$

The coefficients $\varphi^x, \varphi^t, \varphi^{xt}, \varphi^{tt}$ are given in (1.2.7) and the other coefficients of (1.3.20) are obtained by applying the prolongation procedure to differential variables $f\ and\ g$ with independent variables (x,u). In view of (1.3.18), we have

$$\mu^t = \widetilde{D}_t(\mu) - f_x\widetilde{D}_t(\xi^1) - f_u\widetilde{D}_t(\varphi), v^t = \widetilde{D}_t(v) - g_x\widetilde{D}_t(\xi^1) - g_u\widetilde{D}_t(\varphi). \quad (1.3.21)$$

Where $\widetilde{D}_t = \frac{\partial}{\partial t}$. So, we have the following prolongation formulas:

$$\mu^t = \mu t - f_x\xi_t^1 - f_u\varphi_t, \qquad v^t = vt - g_x\xi_t^1 - g_u\varphi_t. \qquad (1.3.22)$$

The invariance conditions (3.19) give rise to

$$\mu^t = v^t = 0, \qquad (1.3.23)$$

that is hold for every $f \ and \ g$. We obtained

$$\mu^t = vt = 0. \qquad \xi_t^1 = \varphi_t = 0.$$

Moreover we obtained

$$\varphi^t = 2f(x,u)u_x\varphi^x = g(x,u)\varphi^{xx} - \mu u_x^2 - vu_{xx} - v = 0. \qquad (1.3.24)$$

The introducing the relation $u_t = f(x,u)u_x^2 + g(x,u)u_{xx}$ to eliminate u_t we are left with a polynomial equation involving the various derivatives of $u(x,t)$ whose coefficients are certain derivatives of $\xi^1, \xi^2, \varphi, \mu,$ and v. We can equate the individual coefficients to zero, leading to the complete set of determining equations:

$$\xi^1 = \xi^1(x) \qquad (1.3.25)$$

$$\xi^2 = \xi^t = 0 \qquad (1.3.26)$$

$$\varphi_{tt} = \xi_t^2 \qquad (1.3.27)$$

$$v = g\xi_t^2 + 2\xi_x^i \qquad (1.3.28)$$

$$\mu = -f\xi_t^2 - f(\varphi_u - 2\xi_x^1) - g\varphi_{uu}\,. \qquad (1.3.29)$$

So, we found that

$$\xi^1(x) = a(x), \qquad \xi^2 = c_1t + c_2, \qquad \varphi(x,u) = c_1u + b(x),$$

$$\mu = -2f\big(c_1 - a(x)\big), \qquad v = -g\big(c_1 - a^t(x)\big), \qquad (1.3.30)$$

with constants c_1, c_2 and two arbitrary functions $a(x) \ and \ b(x) \ such \ that$

$b''(x) = c_1 - a'(x)$. We summarized: The class of Eq. (1.1.2) has an infinite continuous group of equivalence transformations generated by the following infinitesimal operators:

$$Y = a(x)\frac{\partial}{\partial x} + (c_1 t + c_2)\frac{\partial}{\partial t} + \left(c_1 u + b(x)\right)\frac{\partial}{\partial u} - 2f\left(c_1 - a(x)\right)\frac{\partial}{\partial f} - g(c_1 - a'))\frac{\partial}{\partial g}, \qquad (1.3.31)$$

therefore the symmetry algebra of the Burgers' equation (1.1.2) is spanned by the vector fields

$$Y_1 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} - 2f\frac{\partial}{\partial f} - g\frac{\partial}{\partial g}, \quad Y_2 = \frac{\partial}{\partial t},$$

$$Y_3 = a(x)\frac{\partial}{\partial x} + 2fa(x)\frac{\partial}{\partial f} + ga'(x)\frac{\partial}{\partial g}, \quad Y_4 = b(x)\frac{\partial}{\partial u}. \qquad (1.3.32)$$

Table (a): Commutation relations satisfied by infinitesimal generators in (1.4.34)

[,]	Y_1	Y_2	Y_3	Y_4	Y_5
Y_1	Y_1	0	0	0	0
Y_2	Y_1	0	0	Y_2	0
Y_3	Y_1	0	0	Y_3	0
Y_4	Y_1	$-Y_2$	$-Y_3$	0	0
Y_5	Y_1	0	0	0	0

Table (b): Adjoint relations satisfied by infinitesimal generators in (1.4.34)

[,]	Y_1	Y_2	Y_3	Y_4	Y_5
Y_1	Y_1	Y_2	Y_3	Y_4	Y_5
Y_2	Y_1	Y_2	Y_3	$Y_4 - sY_2$	Y_5
Y_3	Y_1	Y_2	Y_3	$Y_4 - sY_3$	Y_5
Y_4	Y_1	$e^{\alpha}Y_2$	$e^{\alpha}Y_3$	Y_4	Y_5
Y_5	Y_1	Y_2	Y_3	Y_4	Y_5

Moreover, in the group of equivalence transformations there are included also discrete transformations, i.e. reflections

$$t \to -t, \quad x \to -x, \quad u \to -u, \quad f \to -f, \quad g \to -g. \qquad (1.3.33)$$

(1.4) Group Classification of Lie Algebras:

One can observe in many applications of group analysis that most of extensions of the principal Lie algebra admitted by the equation under consideration are taken from the equivalence algebra g_s. We call these extensions s-extensions of the principal Lie algebra. The classification of all nonequivalent equations (with respect to a given equivalence group G_s.) admitting s-extensions of the principal Lie algebra is called a preliminary group classification. Here, G_s is not necessary the largest equivalence group but, it can be any subgroup of the group of all equivalence transformations. So, we can take any finite-dimensional subalgebra (desirable as large as possible) of an infinite-dimensional algebra with basis (3.32) and use it for a preliminary group classification. We select the subalgebra g_5 spanned on the following operators:

$$Y_1 = \frac{\partial}{\partial x}, \qquad Y_2 = \frac{\partial}{\partial t}, \qquad Y_3 = \frac{\partial}{\partial u},$$

$$Y_4 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} - 2f\frac{\partial}{\partial f} - g\frac{\partial}{\partial g}. \qquad Y_5 = \frac{\partial}{\partial x} + 2f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g}. \quad (1.4.34)$$

The communication relations between these vector fields were given in table (a). To each s-parameter subgroup there corresponds a funnily of group invariant solutions. So, in general, it is quite impossible to determine all possible group-invariant solutions of a PDE. In order to minimize this search, it is useful to construct the optimal system of solutions. It is well known that the problem of the construction of the optimal system of solutions is equivalent to that of the construction of the optimal system of subalgebras, we will deal with the construction of the optimal system of subalgebras of g_s.

Let G be a Lie group. With g its Lie algebra. Each element $T \in G$ yields inner automorphism $T_a \to TT_aT^{-1}$ of the group G. Every automorphism of the group G induces an automorphism of g. The set of all this automorphism is a Lie group called the adjoint group G^A . The Lie algebra of G^A is the adjoint algebra of g, defined as follows. Let us have two infinitesimal generators $X, Y \in L$. The linear mapping $AdX(Y): Y \to [X, Y]$ is an automorphism of g, called the inner derivation of g. The set of all inner derivations $adX(Y)(X, Y \in)$ together with the Lie bracket $[AdX, AdY] = Ad[X, Y]$ is a Lie algebra g^A called the adjoint algebra of g. Clearly g^A is the Lie algebra of G^A . Two subalgebras in g are conjugate (or similar) if there is a transformation of G^A which takes one subalgebra into the other. The collection of pairwise non-conjugate s-dimensional subalgebras is the optimal system of subalgebras of order s. The construction of one-dimensional optimal system of subalgebras can be carried out by using a global matrix of the adjoint transformations as suggested by L.V.Ovsiannikov (1982). The latter problem tends to determine a list (that is called an optimal system) of conjugacy inequivalent subalgebras with the property that any other subalgebra is equivalent to a unique member of the list under some element of the adjoint representation i.e. $\tilde{h}Ad(g)h$ for some g of a considered Lie group. Thus we will deal with the construction of the optimal system of subalgebras of g_5. The adjoint action is given by the Lie series

$$Ad\big(exp(sY_i)\big)Y_j = Y_j - s\big[Y_i, Y_j\big] + \frac{s^2}{2}\Big[Y_i, \big[Y_i, Y_j\big]\Big] - \cdots, \qquad (1.4.35)$$

where s is a parameter and $i, j = 1, \ldots, 5$. The adjoint representations of g_5 is list in table (b), it consists the separate adjoint actions of each element of g_5 of all other elements.

Theorem (1.4.1): An optimal system of one-dimensional Lie subalgebras of general Burgers' equation (1.1.2) is provided by those generated by

(1) $Y^4 = Y_1 = \partial_t$,

(2) $Y^2 = Y_2 = \partial_x$,

(3) $Y^3 = Y_3 = \partial_u$,

(4) $Y^4 = Y_4 = t\partial_t + u\partial_u - 2f\partial_f - g\partial_g$,

(5) $Y^5 = Y_5 = \partial_x + 2f\partial_f + g\partial_g$,

(6) $Y^6 = Y_1 + Y_2 = \partial_t + \partial_x$,

(7) $Y^7 = -Y_1 + Y_2 = -\partial_t + \partial_x$,

(8) $Y^8 = Y_1 + Y_2 = (t+1)\partial_t + u\partial_u - 2f\partial_f - g\partial_g$,

(9) $Y^9 = -Y_1 + Y_4 = (t-1)\partial_t + u\partial_u - 2f\partial_f - g\partial_g$,

(10) $Y^{10} = Y_1 + Y_5 = \partial_t + \partial_x + 2f\partial_f + g\partial_g$,

(11) $Y^{11} = -Y_1 + Y_5 = -\partial_t + \partial_x + 2f\partial_f + g\partial_g$,

(12) $Y^{12} = Y_4 + Y_5 = t\partial_t + \partial_x + u\partial_u$,

(13) $Y^{13} = -Y_4 + Y_5 = -t\partial_t + \partial_x - u\partial_u + 4f\partial_f + 2g\partial_g$,

(14) $Y^{14} = Y_1 + Y_4 + Y_5 = (t+1)\partial_t + \partial_x + u\partial_u$,

(15) $Y^{15} = Y_1 + Y_4 + Y_5 = (t+1)\partial_t + \partial_x + u\partial_u$,

(16) $Y^{16} = Y_1 - Y_4 + Y_5 = (t+1)\partial_t - u\partial_u + 2f\partial_f + g\partial_g$, (1.4.36)

(17) $Y^{17} = -Y_1 - Y_4 + Y_5 = -(1+t)\partial_t + \partial_x - u\partial_u + 4f\partial_f + 2g\partial_g$,

Proof: Let g_4 is the symmetry algebra of Eq. (1.1.2) with adjoint representation determined in table (b) and

$$Y = a_1Y_1 + a_2Y_2 + a_3Y_3 + a_4Y_4 + a_5Y_5 \qquad (1.4.37)$$

is a nonzero vector field of g. We will simplify as many of the coefficients $a_i, i = 1, \dots, 5$, as possible through proper adjoint applications on Y. We follow our aim in the below easy cases:

case1: At first, we assume that $a_5 \neq 0$. Scaling Y if necessary, also we can assume that $a_5 = 1$ and so we get

$$Y = a_1Y_1 + a_2Y_2 + a_3Y_3 + a_4Y_4 + Y_5. \qquad (1.4.38)$$

Using the table of adjoint (table (b)) . If we act on Y with $Ad(\exp(a_2Y_2))$, the coefficient of Y_2 can be vanished:

$$Y' = a_1Y_1 + a_3Y_3 + a_4Y_4 + Y_5. \qquad (1.4.39)$$

Then we apply $Ad(\exp(a_3Y_3))$ $on\ Y'$ to cancel the coefficient of Y_3:

$$Y'' = a_1Y_1 + a_4Y_4 + Y_5. \qquad (1.4.40)$$

Case 1a: If $a_1, a_4 \neq 0$ then we can make the coefficients of $Y_1\ and\ Y_4\ either + 1\ or - 1$. Thus any one-dimensional subalgebra generated by $Y\ with\ a_3, a_1 \neq 0$ is equivalent to one generated by $\pm Y_1 \pm Y_4 + Y_5$ which introduce parts $(14), (15), (16)\ and\ (17)$ of the theorem.

Case 1*b*: For $a_1 = 0, a_4 \neq 0$ we can see that each one-dimensional subalgebra generated by Y is equivalent to one generated by $\pm Y_4 + Y_5$ which introduce parts (12) and (13) of the theorem.

Case 1c: For $a_1 \neq 0, a_4 = 0$ we can see that each one-dimensional subalgebra generated by Y is equivalent to one generated by $\pm Y_1 + Y_5$ which introduce parts (10) and (11) of the theorem.

Case 2: The remaining one-dimensional subalgebras are spanned by vector fields of the form Y with $a_5 = 0$.

Case 2a: If $a_4 \neq 0$ then by scaling Y, we can assume that $a_4 = 0$. Now by the action of $Ad(expa_2Y_2))$ on Y, we can cancel the coefficient of Y_2:

$$\bar{Y} = a_1Y_1 + a_3Y_3 + Y_4. \qquad (1.4.41)$$

Then by applying $Ad(\exp(a_3Y_3))\ on\ \bar{Y}$ the coefficient of Y_3 can be vanished and we have

$$\tilde{Y}' = a_1Y_1 + Y_4. \qquad (1.4.42)$$

The one-dimensional subalgebra generated by Y is equivalent to one generated by $\pm Y_1 + Y_4$ which introduce parts (8) and (9) of the theorem.

Case 2b: Let $a_4 = 0\ then\ Y\ is\ in\ the\ form$

$$\tilde{Y} = a_1Y_1 + a_2Y_2 + a_3Y_3. \qquad (1.4.43)$$

Suppose that $a_2 \neq 0$ then if necessary we can let it equal to 1 and mew obtain

$$\tilde{Y}' = a_1Y_1 + Y_2 + a_3Y_3. \qquad (1.4.44)$$

By acting $Ad(\exp(a_3Y_3))\ on\ \tilde{Y}', it\ changed\ to\ a_1Y_1 + Y_2$:

Case 2b-1: Let a_1 be nonzero. In this case we can make the coefficient of $Y_1\ in\ \tilde{Y}$ either $+1\ or -1$ and find (6) and (7) sections of the theorem.

Case 2b-2: If a_1 is zero then Y_2 is remained. Hence this case suggests part 2).

Case 2c: Finally if in the latter case a_2 be zero, then no further simplification is possible and then Y is one of cases of (1.4.36). There is not any more possible case for studying and the proof is complete.

The coefficients f, g of Eq. (1.1.2) depend on the variables x, u. Therefore, we take their optimal system's projections on the space (x, u, f, g). The nonzero in $x - axis\ or\ u - axis$ projections of (1.4.36) are:

(1) $Z^1 - Y^2 = Y^6 - Y^7 = \partial_x$,
(2) $Z^2 = Y^3 = \partial_u$,
(3) $Z^3 = Y^4 = Y^8 = Y^9 = -Y^{16} = u\partial_u - 2f\partial_f - g\partial_g$,
(4) $Z^4 = Y^5 = Y^{10} = Y^{11} = \partial_x + 2f\partial_f + g\partial_g$,

(5) $Z^5 = Y^{12} = Y^{14} = Y^{15} = \partial_x + u\partial_u$, (1.4.45)

(6) $Z^6 = Y^{13} = Y^{17} = \partial_x - u\partial_u + 4f\partial_f + 2g\partial_g$.

Proposition (1.4.1): Let $g_m = \langle Y_1, \dots, Y_m \rangle$, be an m-dimensional algebra. Denote by $Y^i(i = 1, \dots, r, 0 < r \leq m, r \in N$ an optimal system of one-dimensional subalgebras of g_m *and by* $Z^i(i = 1, \dots, t, 0 < t \leq r, t \in N)$ the projections of $Y^i. i, e,\ Z^i = pr\left(Y^i\right)$. If equations

$$f = \emptyset(x, u), \qquad g = \psi(x, u), \tag{1.4.46}$$

Table c: The result of the classification

N	Z	Invariant	Equation	Additional operator $X^{(2)}$
1	Z^1	u	u_t	$\partial_x, \partial_t + \partial_x, -\partial_t + \partial_x$
2	Z^2	x	$= \phi u_x^2 + \psi u_{xx}$	∂_u
3	Z^3	x	u_t	$t\partial_t + u\partial_u, (t+1)\partial_t$
4	Z^4	u	$= \phi u_x^2 + \psi u_{xx}$	$+ u\partial_u, (t-1)\partial_t$
5	Z^5	$e^{\frac{u}{x}}$	u_t	$+ u\partial_u$
6	Z^6	$-\frac{1}{u}$	$= u^2 \phi u_x^2$	$\partial_x, \partial_t + \partial_x, -\partial_t + \partial_x$
			$+ e^x \psi u_{xx}$	$t\partial_t + \partial_x + u\partial_u, (t+1)\partial_t + \partial_x$
			u_x	$+ u\partial_u$
			$= e^{x^2} \phi u_x^2$	$-t\partial_t + \partial_x - u\partial_u, -(1+t)\partial_t + \partial_x$
			$+ u\psi u_{xx}$	$- u\partial_u$
			u_t	
			$= \phi u_x^2 + u\psi u_{xx}$	
			u_t	
			$= e^{x^4} \phi u_x^2$	
			$+ e^{x^2} \psi u_{xx}$	

Are invariant with respect to the optimal system Z^i then the equation

$$u_t = \phi(x, u)u_x^2 + \psi(x, u)u_{xx}, \tag{1.4.47}$$

Admits the operators $X^i = projection\ of\ Y^i\ on\ (t, x, u)$.

(5) Conclusion: Finally the classical Lie method and the group classification for the class of Burgers' equation (1.1.2) and investigated the algebraic structure of the symmetry groups for this equation, is obtained. The classification is obtained by constructing an optimal system with the aid propositions (1.4.2). The result of the work is summarized in table c. of course it is also possible to obtain the corresponding reduced equations for all the cases in the classification reported in table c. we omitted these for brevity.

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