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ALGEBRA AND STRUCTURES OF SPINORS FIBER BUNDLES

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المستخلص :

هدف هذه الورقة هو التقصي في رياضيات الحزم التدويمية و تصنيفاتها. ان اسلوب الحزم الليفية الاساسية تسمح من خلال تماسك معالجة شبه متعددات طيّات الريمانيات والتركيبات (البناءات) التدويمية مع جبريات كلايفورد مقترناً بمؤثر ديراك بدراسة تطبيقات عامة في نظرية الكوهومولوجيا .

ABSTRACT.

The aim of this paper is to investigate the mathematics of spinor bundles, and their classification. We devote the methods of principal fiber bundles allows through a coherent treatment of Pseudo-Riemannian manifolds and spinor structures with Clifford algebras which couple to Dirac operator to study important applications in co homology theory.

Introduction :

The interplay between physics and mathematics has spurred each of the disciplines to great heights. It has led to numerous discoveries, not the least of which is the Dirac operator and the related concept of the spinor. The first groundwork for these concepts was laid down by Clifford in the middle of the19th century as a generalization of the quaternions of Hamilton and the exterior algebra[1] of Grassmann. In 1913[1] ´Elie Cartan wrote down the general theory of spinors. Spinors were first applied to mathematical physics by Wolfgang Pauli in 1927[1], when he introduced his spin matrices. On the physics, Dirac introduced his famed operator in 1928[1,2]but made no mention of the connection with spinors, this was only done much lat-

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er. during the years 1940-1970, The subject of spinors and Clifford algebra of bundle became such a fundamental tool of particle physics and came back, later, at the forefront of differential geometry and of mathematics in general, with the recognition of the importance of the Dirac operator and theory of spinors in differential geometry.

All of this now belongs to the standard toolbox of the mathematicians. Spinors may not have yet revealed all their mysteries and they will certainly show up again, one of these days, maybe in a new guise, in Physics and in Mathematics.

This paper will introduce the basic concepts of Clifford algebras, proves a basis for their classification and studied the pin and spin groups as subgroups of the Clifford algebras. The principle of fiber bundles are introduced, two operations which we are interested in, the structure group and constructing spinor fiber bundle (spinor bundle).

Connections on fiber bundles and their relation with covariant derivation is discussed. In the final section, spin structures are introduced as non-trivial coverings of SO-bundles, followed by a proof of necessary and sufficient condition for their existence in terms of the fundamental group of the SO-bundle and Dirac operator.

Clifford algebras

Definition 1:

An associative F-algebra is a triple (A; μ; η), where A is a vector space over the field F, μ: A $\otimes$A $\rightarrow$A and η:$F \rightarrow A$ are linear maps satisfying the conditions[3]

i) (Associativity) $(\mu\otimes id_A) = (id_{A\otimes}\mu)$,

ii) (Unitality) $(\eta \otimes id_A) = id_A = (id_A\otimes\eta)$.

An algebra $A = (A; \mu; \eta)$ is said to be commutative if ab = ba, $\forall\, a, b \in A$.

A Clifford algebra is a type of associative algebra, which can be considered a generalization of the usual associative fields such as $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. In fact, these fields can be seen as particular examples of Clifford algebras[1]. It is defined over a vector space V over a field k equipped with a quadratic form q defined as follows:

Definition 2:

A quadratic form q is an operator from a vector space V to its fields k such that if $\alpha \in k$ and $u, v \in V$ then $q(\alpha v) = \alpha^2 q(v)$ and $2q(u, v) = q(u+v) - q(u) - q(v)$ is a symmetric bilinear form on V [2]. A non-degenerate quadratic form is a quadratic form with the extra condition that $q(v) = 0 \Leftrightarrow v = 0$[1].

A Clifford algebra is a generalization of the exterior or Grassmann algebra. Precisely:

Definition 3:

Let V be a vector field over a field k, and take the tensor algebra $T(V) = \sum_{i=0}^{\infty} \bigoplus^{i} V = k \oplus V \oplus V \otimes V \oplus ...$ and let q be a quadratic form[1].

A Clifford algebra $C\ell(V, q)$ on V is then $C\ell(V, q) = T(V)/I(V, q)$ with $I(V, q)$ the two-sided ideal generated by $v \otimes v + q(v)1$[1,2]. This is an associative algebra with unit. The exterior algebra can be identified with the Clifford algebra with quadratic form 0. In defining the Clifford algebra and deriving its basic properties, we follow[2], but we could have equivalently started with the universal property proven below and worked our way back from there.

We can define a natural embedding i from V to $C\ell(V, q)$ by composing with the natural embedding of V in the tensor algebra: $i_q : V \rightarrow T(V) \rightarrow C\ell(V, q)$,(see[1]).

Lemma 4:

The embedding $i_q|V : V \longrightarrow C\ell(V, q)$ is injective.

Clifford algebras have also a universal property, which could also be used to define them.

Lemma 5:

Every vector space with a quadratic form has a q-orthogonal basis $\{e_1, \dots, e_n\}$, that is, $q(e_i, e_j) = 0$ if $i \neq j$ [1].

Definition 6:

Let k be a field. It is called a spin field if for all $a \in k$ either the equation $x^2 = a$ or $x^2 = -a$ has a solution[1]. So we can look at composition of $C\ell(V, q)$ into eigenspaces of

this operator, $C\ell(V, q) = C\ell^0(V, q) \oplus C\ell^1(V, q)$, where $C\ell^0(V, q)$ is the even part and $C\ell^1(V, q)$ the odd part and we have $C\ell^i(V, q)$. $C\ell^j(V, q) \subseteq C\ell^{i+j}(V, q)$ with $i+j$ taken mod 2. This is called a $\mathbb{Z}_2$-graded algebra.

Definition 7:(Pin and Spin groups)

The group Pin is the following subgroup of the Clifford group:

$Pin(V, q) = \{v \in \check{P}(V, q) \mid q(v) = \pm 1\}$,*i,e*; we define $P(V,q) \subset \check{P}(V, q)$, (where $\check{P}(V, q)$ calls *Lipschitz group* which is“largest” spinor group) to be the subgroup of $C\ell(V, q)$ generated by the elements $v \in V$ with $q(v) \neq 0$. and that there is a representation

$$P(V,q)\xrightarrow{Ad}O(V,q)$$

The group Spin is the even subgroup of Pin: $Spin(V, q) = Pin(V, q) \cap C\ell^0(V, q)$[1,2].

Theorem 8:

Let k be a spin field. Then $Pin(V, q)$ is the kernel of $N\colon \check{P}(V, q) \to k^\times$, (*where* $k^\times$ *is nonzero multiples of* 1) and the twisted adjoint $\widetilde{Ad}|_{Pin(V, q)}$ is a surjection of $Pin(V, q)$ onto $O(V, q)$. We have the exact sequence, see[1].

$$1\to\mu_4(\mathrm{k})\to\mathrm{Pin}(\mathrm{V},\ \mathrm{q})\xrightarrow{\tilde{Ad}}\mathrm{O}(\mathrm{V},\ \mathrm{q})\to1$$

with $\mu_4(k)$ the 4th roots of 1 in the field. For example, $\mu_4(\mathbb{R}) = \{-1, 1\}\approx \mathbb{Z}_2$.

And $\mu_4(\mathbb{C}) = \{-1, 1, -i, i\} \approx \mathbb{Z}_4$[1,10].

We can also consider the covering group of $SO(V, q) = \{x \in O(V, q)| \det(x) =1\}$. Then we have

Corollary 9:

The group $Spin(V, q)$ is a cover of $SO(V, q)$, and we have the exact sequence:

$$1\to\mu_4(\mathrm{k})\to Spin(\mathrm{V},\ \mathrm{q})\xrightarrow{\tilde{Ad}}\mathrm{SO}(\mathrm{V},\ \mathrm{q})\to1$$

with $\mu_4(k)$ as in theorem 3 [1,10].

Proof: Let $x\in Pin(V, q)$. Then $\widetilde{Ad}_x$ is equal to the composition of a number of reflections due to theorem 8. We have that for all $v\in V$, $\det(\widetilde{Ad}_v) = -1$. To see this, take a q-orthogonal basis with $v_1= v$ and $q(v, v_j) = 0$ for all $j >1$. Then $\widetilde{Ad}_v(v_1) = -v$ by definition and $\widetilde{Ad}_v(v_j) = v_j$ with $j >1$ by definition, so $\det(\widetilde{Ad}_v) = -1$. So any element of $SO(V, q)$ is generated by an even number of reflection, and so $Spin(V, q)$ must be generated by an even number of vectors, so by the properties of the $\mathbb{Z}_2$ grading we get that it is also an element of $C\ell^0(V, q)$. The exact sequence follows immediately from theorem 8[1].

Proposition 10:

For any x in Spin(n), there exists an even integer p and elements $f_1,\ldots, f_p$ of norm 1 such that $x = f_1\ldots f_p$. The reverse statement also holds[12].

Theorem 11:

For any $n \geq 2$, $Spin(n)$ is connected. For $n\geq3$, $Spin(n)$ is simply connected, hence $Spin(n)$ is the universal covering Lie group of $SO(n)$.

Proof. We have a short exact sequence of Lie groups

$$1\to\mu_4(\mathrm{k})\to Spin(\mathrm{V},\ \mathrm{q})\xrightarrow{\tilde{Ad}}\mathrm{SO}(\mathrm{V},\ \mathrm{q})\to1$$

Since $\pi_1 SO(n) = \mathbb{Z}_2$ for any $n \geq 3$, any connected double covering of $SO(n)$ is simply connected for $n \geq 3$. Hence it suffices to show that $Spin(n)$ is connected for any $n \geq 2$ [10].

Firstly, let us show that 1 and -1 are path-connected in $Spin(n)$. Since the dimension is at least 2, there are elements e_1 and e_2 in V such that $\|e_1\| = \|e_2\| = 1$ and $\langle e_1, e_2 \rangle = 0$. For $0 \leq t \leq \pi$, we define $\gamma(t) = \cos t + e_1 e_2 \sin t = e_1(-e_1 \cos t + e_2 \sin t)$. It is trivial that the norm $(-e_1 \cos t + e_2 \sin t)$ is 1. By Proposition10, $\gamma(t)$ is an element of $Spin(n)$ for any t. We have $\gamma(0) = 1$ and $\gamma(\pi) = -1$.

Secondly, note that any element y of $Spin(n)$ can be connected with $-y$ by the path $y\gamma(t)$.

Lastly, let x and y be in $Spin(n)$. Since $SO(n)$ is connected, there exists a path from $\rho(x)$ to $\rho(y)$ in $SO(n)$. We may lift this path to a path starting in x and ending in a point y' of Spin(n). Since the kernel of ρ equals $\{\pm 1\}$, it holds that $y' = y$ or $y' = -y$. In the first case we are done. In the second case, we connect x with $-y$ and then connect $-y$ with y [12].

We now can construct any real Clifford algebra we are interested in. For example, the physically interesting Clifford algebras $C\ell_{1,3}$ and $C\ell_{3,1}$ are now constructed as $C\ell_{1,3} \simeq C\ell_{0,2} \otimes C\ell_{1,1} = \mathbb{H} \otimes \mathbb{R}(2) \simeq \mathbb{H}(2)$ and $C\ell_{3,1} \simeq C\ell_{2,0} \otimes C\ell_{1,1} = \mathbb{R}(2) \otimes \mathbb{R}(2) \simeq \mathbb{R}(4)$ [1].

Theorem 12:

There are isomorphisms

$$\begin{gathered} C\ell_{n,0} \otimes C\ell_{0,2} \simeq C\ell_{0,n+2} \\ C\ell_{0,n} \otimes C\ell_{2,0} \simeq C\ell_{n+2,0} \\ C\ell_{r,s} \otimes C\ell_{1,1} \simeq C\ell_{r+1,s+1} \end{gathered}$$

or all $n, r, s \geq 0$[2].

Note that here we are using the ungraded tensor product[1].

Remark13: It is standard notation to write: $q_{r,s} \equiv q$, $O_{r.s} \equiv O(V, q)$ and $SO_{r,s} \equiv SO(V,q)$. In accordance we write $Pin_{r,s} \equiv Pin(V, q)$ and $Spin_{r,s} \equiv Spin(V, q)$. Similarly, it is conventional to write $O_n \equiv O_{n,0} \cong O_{0,n}$ and $SO_n \equiv SO_{n,0} \cong SO_{0,n}$,Thus, we set $Pin_n = pin_{n,0}$, and $Spin_n = Spin_{n,0}$.

We also write $P_{r.s} \equiv P(V, q)$ and $\check{P}_{r,s} \equiv \check{P}(V, q)$[2].

Definition 14:

A sections of Clifford $C\ell(V, q)$ *are called Clifford field* [5].

Spinors

We will be constructing spinor m- dimension representations on complex vector, the complex Clifford algebras will turn out to have a much simpler structure than the real ones, with a periodicity of degree 2 rather than degree 8 as in the real case[10]. Now if m is even, $m = 2n$ we have *Weyl* representation which restricts to $C\ell^0(V, q)$ and decomposes into the direct sum of two complex in equivalent, *half-spinor* (σ_+ and σ_-). If m is odd, $m = 2n - 1$, then the algebra $C\ell^1(V, q)$ is central simple ;it has a faithful and irreducible *Pauli* representation in a complex vector space S_0. extends to two complex in equivalent representations σ and $\sigma{\circ}\alpha$ of the full algebra $C\ell\ (V, q)$ such that $\sigma(\eta) = \iota(q)id_{S_0}$, where $\imath(q)\in\{1, \sqrt{-1}\}$ so that $\eta^2 = \imath(q)^2$. The *Cartan* representation $\sigma\bigoplus\sigma{\circ}\alpha$ of $C\ell(V, q)$ in the 2^n-dimensional vector space $S= S_0\bigoplus S_0$ is faithful. The *Dirac* operator on odd-dimensional, non-orientable pin manifolds, is relates to name of Pauli are associated by physicists with spinors in dimensions 4 and 3, respectively. [15].

Proposition 15:

Consider the sequence of homomorphisms of algebras

$$C\ell_m \xrightarrow{inj} C\ell_{m+1} \xrightarrow{i_{m+1}} C\ell^0_{m+2} \xrightarrow{\theta} EndS$$

If m is *even* (resp., *odd*) and θ is a *Weyl* (resp., *Pauli*) representation, then $\theta{\circ}i_{m+1}$ is a *Pauli* (resp., *Dirac*) representation and $\theta{\circ}i_{m+1}{\circ}inj$ is a *Dirac* (resp., *Cartan*) representation[15].

Definition 16:

Any representation of $C\ell(V, q)$ or $C\ell^0(V, q)$ equivalent to one of the representations described in proposition 15 is called a spinor representation of that algebra[15].

Definition 17:

The complex Clifford algebra $C\ell^{\mathbb{C}}(\mathrm{V}, q)$ is the Clifford algebra constructed by starting with the complexified vector space $V\otimes_{\mathbb{R}}\mathbb{C}$, extending q to this by complex-linearity, then using the same definition as in the real case. If we start with a real vector space V of dimension n, this will be denoted by $C\ell^{\mathbb{C}}(n)$.

One can easily see that $C\ell^{\mathbb{C}}(n) = C\ell(n)\otimes C\ell.$ The construction of the spin representation as invertible elements in $C\ell(n)$ can also be complexified, producing a construction of Spin(n, $\mathbb{C}$) (the complexification of Spin(n)) is an invertible elements in $C\ell^{\mathbb{C}}(n)$.

We will study the structure of the algebras $C\ell^{\mathbb{C}}(n)$ by an inductive argument.

To begin the induction, recall that

$$C\ell(1) = \mathbb{C},\ C\ell(2) = \mathbb{H}$$

so

$$C\ell^{\mathbb{C}}(1) = C\ell(1)\otimes_{\mathbb{R}}\mathbb{C} = \mathbb{C}\oplus\mathbb{C}$$

and

$C\ell^{\mathbb{C}}(2) = C\ell(2)\otimes_{\mathbb{R}}\mathbb{C} = \mathbb{H}\oplus_{\mathbb{R}}\mathbb{C} = M(2, \mathbb{C})$

Theorem 18:

$C\ell^{\mathbb{C}}(n+2) = C\ell^{\mathbb{C}}(n)\otimes_{\mathbb{C}}C\ell^{\mathbb{C}}(2) = C\ell^{\mathbb{C}}(\mathrm{n})\otimes_{\mathbb{C}}M(2, \mathbb{C})$.

Using the cases $n = 1, 2$ to start the induction, one finds

Corollary 19:

If $n = 2k$, $C\ell^{\mathbb{C}}(2k) = M(2, \mathbb{C})\otimes \cdot\cdot\cdot \otimes M(2, \mathbb{C}) = M(2^K, \mathbb{C})$ where the product has k factors, and if $n = 2k +1$, $C\ell^{\mathbb{C}}(2k +1) = C\ell^{\mathbb{C}}(1)\otimes M(2^k, \mathbb{C}) = M(2^K, \mathbb{C})\oplus M(2^K, \mathbb{C})$[10].

Proof: Choose generators h_1, h_2 of $C\ell(2)$, $f_1 ,\ldots, f_{\mathrm{n}}$ of $C\ell(n)$ and $e_1, \cdot\cdot\cdot e_{\mathrm{n+2}}$ of $C\ell(n +2)$. Then the isomorphism of the theorem is given by the following map of generators

$$e_1 \to 1\otimes h_1$$
$$e_2 \to 1\otimes h_2$$
$$e_3 \to if_1\otimes h_1h_2$$
$$\cdots$$
$$e_{n+2} \to if_n\otimes h_1h_2$$

One can check that this map preserves the Clifford algebra relations and is surjective, thus an isomorphism of algebras.

From now we'll concentrate on the even case $n = 2k$. In this case we have seen that the complexified Clifford algebra is the algebra of 2^k by 2^k complex matrices. A spinor space S will be a vector space that these matrices act on.

Definition 20:

A spinor module S for the Clifford algebra $C\ell^{\mathbb{C}}(2k)$ is given by a choice of a 2^k-dimensional complex vector space S, together with an identification $C\ell^{\mathbb{C}}(2k) = End(S)$,by anther words $C\ell(K)\oplus_{\mathbb{R}}\mathbb{C}) = End(S^K)$ of the Clifford algebra with the algebra of linear endomorphisms of S.

So a spinor space is a complex dimensional vector space S, together with a choice of how the $2k$ generators e_i of the Clifford algebra act as linear operators on S[9], to actually construct such an S , together with appropriate operators on it,

we will use
exterior algebra techniques.

Definition 21:

Let P, X, F be manifolds. A map $\pi{:}P\rightarrow X$ is called a fiber bundle with fiber F when there is an open covering U_α of X such that for every U_α we have that there is a diffeomorphism from $\varphi_\alpha{:}\pi^{-1}(U_\alpha)\rightarrow U_\alpha\times F$, such that $\pi\varphi_\alpha=\mathrm{Id}_{U\alpha}$. This φ_α is called a local trivialization of P , and π is called projection[14].

A principal fiber bundle is defined such that the fibers are not only manifolds, but also groups, and elements of the group act in a nice way on the fiber bundle.

Take G a Lie group, and P, X manifolds. A *G-principal fiber bundle* is a fiber bundle $\pi : P\rightarrow X$, where G acts freely on the right of P , and the orbits of G are precisely the fibers, so there is an open covering U_α of X such that $\forall_\alpha{:}\pi^{-1}(U_\alpha)$ is diffeomorphic to $U_\alpha\times G$.

Let φ be a homomorphism of groups, that is, a group-structure preserving map, from G' to G[7]. A homomorphism of principal fiber bundles f between a G'-principal
fiber bundle P' over M' and a G-principal fiber bundle P over M is a map such that
f: $P^{'}\rightarrow P$ is smooth and $f(pg) = f(p)\varphi(g)$ for all $p\in P'$ and $g\in G'$. Obviously, there is then an induced mapping $\bar{f}$ such that the following diagram commutes:

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\pi'"'] & P' \arrow[d, "\pi"] \\ M \arrow[r, "\bar{f}"'] & M' \end{tikzcd}$$

Two principal fiber bundles P and P' are said to be equivalent if there is a homomorphism $H{:}P\rightarrow P'$ such that $H(pg) = H(p)g$ for all $g\in G$ and $p\in P$.

An important construction in the theory of fiber bundles is the associated bundle. Take a principal fiber bundle P over a manifold X .Consider a manifold F with all smooth diffeomorphisms Diffeo(F) with the compact-open topology, that is, take as basis for the topology finite intersections of $B(K,U)$, the sets of functions that map a compact set K into an open set U . Now take a continuous Homomorphism $\rho{:}G\rightarrow\mathrm{Diffeo}(F)$, and we can construct a fiber bundle over X as follows:

Define an equivalence relation by the free left action of G on the product $P\times F$ given by $\varphi g\,(p, f) = (pg^{-1}, \rho(g)f$.

Definition 22:

Define $P\times_{\rho}F$ to be the quotient space of this equivalence relation. Clearly, the projection $P\times F \longrightarrow P \xrightarrow{\pi} X$ descends to a mapping $\pi_{\rho}: P\times_{\rho}F \longrightarrow X$. $P\times_{\rho}F$ is called the bundle associated to P by ρ.

A special case of a bundle associated to a fiber bundle is a vector bundle E associated to a principal $SO(r, s)$-bundle. The group $SO(r, s)$ then acts on the fibers of the vector bundle as an orthonormal, orientation preserving change of basis, and is called the orthonormal frame bundle of E, or $F(E)$[1].

Clifford and spinor bundles over Spinor structure on manifolds

Recall that $\mathrm{Spin}_{(r,s)}$ is a 2-sheeted covering of $SO(r, s)$, so it is natural to look at when there exists a 2-sheeted covering of a principal $SO(r, s)$ bundle (orthonormal bundle) such that it is a principal $Spin_{(r,s)}$ bundle. This is called a spin structure[2,7,8,16]. the theory of spinor structures for Pseudo-Riemannian manifolds can be subsumed into the general discussion that we give here. We will see also that the correspondence between covariant derivatives and connections on an orthonormal bundle fits into this theory.

Definition 23:

A spin structure on a Pseudo-Riemannian vector bundle with signature(r, s) E is a principal $Spin(r, s)$ bundle $P_{Spin}(E)$ together with a 2- sheeted covering $\xi : P_{Spin}(E) \longrightarrow P_{SO}(E)$ such that $\xi(pg) = \xi(p)\xi_0(g)$ for all $p\in P_{Spin}$ and $g\in Spin(r, s)$ and ξ_0 the covering map $Spin(r, s) \rightarrow SO(r, s)$[2,10].

Two spin structures $P_{Spin}(E)$ and $P'_{Spin}(E)$ are called equivalent if there is a mapping F such that $F(gh) = F(g)h$ for $g\in P_{Spin}$ and $h\in Spin$ and the following diagram commutes:

$$\begin{tikzcd}
P_{Spin}(E) \arrow[r, "F"] \arrow[dr, "\xi"'] & P'_{Spin}(E) \arrow[d, "\xi'"] \\
& P_{SO}(E)
\end{tikzcd}$$

This means that if they are equivalent as spin structures, they are equivalent as principal fiber bundles[1,2].

From now on, we will only consider r, s such that $\pi_1(SO(r, s)) = \mathbb{Z}_2$. This makes many proofs much simpler, since it makes Spin$_{(r,s)}$ simply connected. Also, this situation is the one where most examples interesting to physists are, when $r \geq 3$ and $s = 0$ or 1, or vice-versa. The definition of the spin structure gives the following commutative diagram

$$\begin{tikzcd}
P_{Spin}(E) \arrow[rr, "\xi"] \arrow[dr, "\pi'"'] & & P'_{SO}(E) \arrow[dl, "\pi"] \\
& X &
\end{tikzcd}$$

Since restriction to fibers gives the covering map ξ_0 , this diagram can be extended[1,2]:

$$\begin{tikzcd}
& Spin(r,s) \arrow[r, "\xi_0"] \arrow[d] & SO(r,s) \arrow[d] \\
\mathbb{Z}_2 \arrow[ur] \arrow[r] & P_{Spin}(E) \arrow[r, "\xi"] \arrow[d, "\pi'"'] & P_{SO}(E) \arrow[dl, "\pi"] \\
& X &
\end{tikzcd}$$

With the vertical lines the inclusions of fibers in the fiber bundle. Now we can consider whether given a 2-sheeted covering $\varphi : C_2(E) \to P_{SO}(E)$ of $P_{SO}(E)$ gives a spin structure. It certainly gives a fiber bundle over E, since we can set $\pi' = \pi \circ \xi$.

Now we see that this bundle gives a spin structure if the covering is non-trivial on the fibers. Hence we have;

Theorem 24:

If $\pi_1(SO(r, s)) = \mathbb{Z}_2$ then the spin structures are in one-to-one correspondence with 2-sheeted coverings of P_{SO}(E) which are non-trivial on the fibers[1,2].

Now consider a spin structure $\xi : P_{Spin} \to P_{SO}$ .Define $\alpha_F \in \pi_1(SO\ (r,s))$ the non-trivial element. The spin structure ξ induces a group homomorphism $\xi_* : \pi_1(P) \to \pi_1(Q)$. This subgroup of $\pi_1(Q)$ is a subgroup of index 2, because of covering morphisom and P_{Spin} is a double covering of P_{SO} fiberwise.

Lemma 25:

$\alpha_F \notin \xi_*(\pi_1(P_{Spin}))$.

Proof: Suppose $\alpha_F \in \xi_*(\pi_1(P_{Spin}))$. Then the inclusion map $i{:}SO \to P_{SO}$ lifts to a continuous map

$I{:}SO \to P_{Spin}$ such that

$$\begin{tikzcd}
 & P_{Spin} \arrow[dd] \\
SO \arrow[ru, "I"] \arrow[rd, "i"'] & \\
 & P_{SO}
\end{tikzcd}$$

Commutes. $I\,(SO) \subset P_{Spin}$ is contained in one fiber Spin of P_{Spin} , so $I{:}SO(r,s) \to Spin_{(r,s)}$ with $\xi_0 \circ I = \mathrm{Id}SO(r,s) = \mathrm{Id}\mathbb{Z}_2$. Then $\xi_{0*}I_* = \mathrm{Id}\pi_1(SO(r,s))$ and $\pi_1(SO(r,s)) = \mathbb{Z}_2$ and $\pi_1(Spin(r,s)) = 1$ so we get a contradiction, so $\alpha_F \notin \xi_*(\pi_1(P_{Spin}))$[7].

This lemma is then used to prove a classification theorem of spin structures, following and using the classification of covering spaces.

Theorem26:

A $SO(r,s)$-principal fiber bundle P_{SO} over a manifold M has a spin structure if and only if there is a short split exact sequence.

$$0 \longrightarrow \mathbf{Z}_2 \xrightarrow{i_*} \pi_1(P_{SO}) \xrightarrow{\pi_*} \pi_1(M) \longrightarrow 0$$

meaning that $\pi_1(P_{SO})$ is isomorphic to $K \times \mathbb{Z}_2$ and π_*, the map induced by projection map, maps K isomorphically to $\pi_1(M)$.

To prove this theorem, we need a few lemmas. First we prove the necessity of the conditions, we assume P_{SO} has a spin structure P_{Spin} .

Lemma 27:

The map $\pi'_*{:}\pi_1(P_{Spin}) \to \pi_1(M)$ is an isomorphism

Proof. First we prove subjectivity. Take an element $[g] \in \pi_1(M)$ represented by a loop $g:[0,1] \to M$ based at $m_0 \in M$. We can then look at a path $\tilde{g}:[0,1] \to P_{Spin}$ such that $\pi\tilde{g}(t) = g(t)$. This path exists, because P_{Spin} is a fiber bundle over M. This path does not have to be a loop. However, $\pi(\tilde{g}(0)) = \pi(\tilde{g}(1)) = m_0$ and the fibers of P_{Spin} are path-connected, so we can find a path δ in Spin such that $\delta(0) = 1$ and $\delta(1)\tilde{g}(1) = g(0)$. The product path $\delta\tilde{g}$ is now a loop and $\pi(\delta\tilde{g}) = g$. This proves

subjectivity.

As for infectivity, consider a loop $\widetilde{h}$: $[0, 1] \to P_{Spin}$, and assume $\pi(\widetilde{h})(t) = h(t)$ is hemitropic to trivial loop, say at the point $x_0 = h(0)$. There is then a homogony from the loop h to the point x_0. This homogony can be covered by a homogony of $\widetilde{h}$ into a new loop lying in the fiber $\pi^{-1}(x_0)$, or its generalization to fiber bundles over par compact spaces)[13]. Since Spin is simply connected, there is then a homogony to a single point in this fiber, hence $\widetilde{h}$ is hemitropic to the trivial loop, so π_* is

injective.

Lemma 28:

The map $\pi_*{:}\pi_1(P_{SO}) \to \pi_1(M)$ restricted to the image of $\xi_*(\pi_1(P_{Spin}))$ is an isomorphism.

Proof: Call the image of ξ_*K. We have the following commutative diagram:

$$\begin{tikzcd}
\pi_1(P_{Spin}) \arrow[r, "\xi_*"] \arrow[d, "\pi_*"] & \pi_1(P_{SO}) \arrow[dl, "\pi_*"] \\
\pi_1(M) &
\end{tikzcd}$$

The map ξ_* is injective, according to the covering space morphism is injective. We

also know that π' is an isomorphism, so $\pi_{*|}k$ is injective. Furthermore $\pi_*|k \circ \xi_* = \pi'$ is

an isomorphism, so $\pi_*|k$ must be surjective, hence an isomorphism[1].□

Lemma 29:

The map $i_*{:}\pi_1(SO) = \mathbb{Z}_2 \to \pi_1(P_{SO})$ is injective, so it is an isomorphism onto its image.

Proof: Suppose $[g] \in \pi_1(SO)$ and $i_*[g] = [\alpha] \in \pi_1(P_{SO})$. If $[\alpha]$ is trivial, then we can lift α to a path $\hat{\alpha}$ in P_{Spin} .The loop α lies within a single fiber, so the loop $\hat{\alpha}$ also does. Because Spin is simply connected, there is a homotopy between $\hat{\alpha}$ and the trivial loop within the fiber. Applying the covering map to of P_{SO}, SO, so $[g]$ is trivial and i_* is injective.□

Lemma 30:

The sequence of lemma27 is exact, in particular: $ker(\pi_*) = img(i_*)$[1,2].

Proof : Take a $[g] \in \pi_1(SO)$. Then $i_*[g]$ has a representant lying within a single fiber. Then $\pi_* i_*[g] = [e]$, hence $\mathrm{im}(i_*) \subseteq \ker(\pi_*)$. Take a

loop $\alpha \subseteq P_{SO}$. We will construct a loop $\hat{\alpha} \subset i(P_{\mathrm{Spin}})$ such that $\alpha = \hat{\alpha} g$ with g a loop in SO. First let $\alpha' = \pi_\circ \alpha$ be a loop in M. Then using δ, we lift this loop one in P_{Spin}, by lifting and multiplying with a path δ with $\delta(0) = 1$ and $\alpha'(1)\delta(1) = \alpha(0)$. Now define $\hat{\alpha} = \xi(\alpha'\delta)$. We can now conclude that
$[\hat{\alpha}] = \xi_* \pi'^{-1}_* \pi_*([\alpha])$ and that $\pi(\alpha) = \pi(\hat{\alpha})$. Thus we have that $\alpha = \hat{\alpha} g$ for some loop
$g \subseteq SO$, hence $[\alpha] = [\hat{\alpha} g] = [\hat{\alpha}] i_*[g]$, so if $[\alpha] \in \ker(\pi_*)$ we have that because π_* is an isomorphism on the image of ξ_*, that $[\alpha] = [\hat{\alpha}] i_*[g] = [e] i_*[g]$, so $\ker(\pi_*) \subseteq \mathrm{im}(i_*)$[1]. □

If the fundamental group of $SO(r, s)$ is not $\mathbb{Z}_2$ this result can be generalized, but the double covering space of $SO(r, s)$ will not be a universal covering space.

Instead, one must look at either the universal covering space of $SO(r, s)$ which is then not equal to $\mathrm{Spin}_{(r,\, s)}$, or one must look at not simply connected $\mathrm{Spin}_{(r,\, s)}$[1,7].

The condition of lemma27 for the existence of a spin structure can be shown to be equivalent with the usual condition that the second Stiefel-Whitney class w_2 of M vanishes[2,7,17]. For the special case of $SO_0(1,3)$-principal fiber bundles over a noncompact 4-manifold M , the case in general relativity, it has been shown that any existing spin structures are trivial, $M \times Spin(1, 3)$ and so it has a spin structure, if and only if the $SO_0(1,3)$- bundle is parallelizable, meaning that there is
a global section of the $SO_0(1, 3)$ bundle[1].

Definition 31:

We called global sections of $P_{SO(1,\,3)}(M)$ Lorentz frames and global sections of $P_{Spin\,(1,\,3)}(M)$ spin frames[5].

Remark 32. When we will use the concept of spin structure in physics, the space-time is described as a four-dimensional manifold endowed with a metric of signature (3, 1) or (1, 3), we will naturally use $SL(2, \mathbb{C})$ as group instead of $\mathrm{Spin}(p, q)$.

The isomorphism $SL(2, \mathbb{C}) \simeq \mathrm{Spin}(1, 3)$[8]. Here the fundamental interactions of spinors, Dirac spinors and Wely spinors (half-spinors) are used to describe the most fundamental particles. Majorana spinors and biquaternions, although mathematically available in same dimension.

Clifford algebras and spinor bundles

There are two equivalent ways of defining a Clifford bundle of a Pseudo-Riemannian vector bundle E over a manifold X. One is the obvious generalization of $C\ell(\mathbb{R}^n)$:

$$C\ell(E) = \bigcup_{x\in X} C\ell(E_x, q_x) \qquad x \in X$$

With q a smooth quadratic form on E and q_x the restriction of that form to the fiber over x. This definition emphasizes that the Clifford bundle is a bundle of Clifford

algebra's over X. The other definition uses associated bundles, and can be used to

determine the topology, as follows: An orthogonal transformation with respect to an inner product of signature (r,s) in $\mathbb{R}^{(r+s)}$ induces an orthogonal transformation in $C\ell_{r,s}$, since it preserves the ideal. This induced map preserves the multiplication in $C\ell_{r,s}$, so if we take an orthogonal transformation $\rho_{r,s}$ from $SO(r, s)$, we get a map $cl(\rho_{r,s}):SO_{r,s} \to \mathrm{Aut}(C\ell_{r,s})$. The bundle associated to this bundle is called the Clifford bundle:

$C\ell(E) = P_{SO}(E)\times_{cl(\rho r,s)} C\ell_{r,s}$

In fact $C\ell(E)$ could be defined as the quotient bundle:

$C\ell(E) = \left(\sum_{r=0}^{\infty} \otimes^r E\right)/I(E)$

where $I(E)$ is the bundle of ideals[2].

It is also evident that each of the notions intrinsic to Clifford algebras carries over to Clifford bundles. For example, there is a decomposition

$C\ell(E) = C\ell^0(E) \oplus C\ell^1(E)$

corresponding to the even-odd decomposition of the algebras.These are the +1 and -1

eigenbundles of the bundle automorphism $\alpha: C\ell(E) \to C\ell(E)$ [10].

These two definitions are the same, since the fiber at $x \in E$ of $P_{SO}(E)\times_{cl(\rho r,s)} C\ell_{r,s}$ is $C\ell_{r,s}$[14]. All notions familiar from Clifford algebras over real vector spaces carry over to Clifford bundles over manifolds. If X is a Pseudo-Riemannian manifold, we can construct the Clifford bundle $C\ell(TX)$ associated with the Pseudo-Riemannian form on the tangent bundle TX. We will also call this bundle $C\ell(X)$, in case there is no confusion possible[14].

Definition 33:

An oriented manifold endowed with a spin structure will be called a spin manifold[5].

We are now going to define geometric differential operators that are closely connected with the topological or geometrical structure of an oriented Riemannian manifold X.

Definition 34:

Let X be an oriented manifold with a spin structure $\xi : P_{SPin}(X) \to P_{so}(X)$. A real or complex spinor bundle of X is a bundle of the form[1,2,10]

$$S(X) = \begin{cases} P_{spin}(X) \times_{\mu} L \\ P_{spin}(X) \times_{\mu} L_{c} \end{cases}$$

with L a left module of $C\ell_{r,s}$ and $\mu : Spin(r, s) \to End(L)$ is the representation given by left-multiplication of elements in $Spin(r, s)$, and where $L_{\mathbb{C}}$ is a complex left module for $C\ell(\mathbb{R}^{n}) \otimes \mathbb{C}$ and $\mu : Spin(r, s) \to End(L_{\mathbb{C}})$ is the representation given by left-multiplication of elements in $Spin(r, s)$. When $(r, s) = (1, 3)$were defined as certain equivalence classes in appropriate sets, and a preliminary definition for field of these object living on living *Minkowski* space-time was given[5].

Remark 35: In what follows we denote the complexified left spin Clifford bundle by $C\ell^{l}_{SPin_{1,3}}(X) = P_{SPin_{1,3}}(X) \times_{l} \mathbb{C} \otimes \mathbb{R}_{1,3} \equiv P_{SPin_{1,3}}(X) \times_{l} \mathbb{R}_{4,1}$ and the complexified right Clifford bundle by $C\ell^{r}_{SPin_{1,3}}(X) = P_{SPin_{1,3}}(X) \times_{r} \mathbb{C} \otimes \mathbb{R}_{1,3} \equiv P_{SPin_{1,3}}(X) \times_{r} \mathbb{R}_{4,1}$[5].

Remark 36: Taking e.g., $L_{\mathbb{C}} = \mathbb{C}^{4}$ and $\mu_{\mathbb{C}}$ the $D^{(1\backslash 2,0)} \oplus D^{(O,1\backslash 2)}$ of $Spin_{1,3} \cong SL(2, \mathbb{C})$ in $End(\mathbb{C}^{4})$, we immediately recognize the usual definition of the covariant spinor bundle of L[5],(*where D is Dirac operator see the last sections*).

If the module L(or $L_{\mathbb{C}}$)is $\mathbb{Z}_{2}$-graded, the corresponding bundle is said to be $\mathbb{Z}_{2}$-graded.

Example37: Consider $C\ell(\mathbb{R}^{n})$ as a module over itself by left multiplication l.

The corresponding real spinor bundle $C\ell_{SPin}(X) = P_{SPin}(X) \times_{l} C\ell(\mathbb{R}^{n})$ *,then*:

(i)$C\ell_{\mathrm{SPin}}(X)$ is a "principal $C\ell(\mathbb{R}^{n})$-bundle", i.e., it admits a free action of $C\ell(\mathbb{R}^{n})$ on the right.

(ii)There is a natural embedding $P_{SPin}(X) \subset C\ell_{spin}(X)$ which comes from the embedding $\mathrm{Spin}_{n} \subset C\ell(\mathbb{R}^{n})$. Hence, every real spinor bundle for X can be captured from this one[2].

A similar remark holds for the complex case.

Of course, the bundle $C\ell_{\mathrm{SPin}}(X)$ differs from the Clifford bundle $C\ell(X)$. They can be compared as follows. Consider the representation

$$\mathrm{Ad}: Spin^{n} \longrightarrow Aut(C\ell(\mathbb{R}^{n}))$$

given by $\mathrm{Ad}_{g}(\varphi) = g\varphi g^{-1}$ for $g \in Spin^{n} \subset (C\ell(\mathbb{R}^{n}))$. Clearly Ad_{-1} = identity, and so this representation descends to a representation Ad' of SO_{n}. One easily checks that Ad' is just the representation $C\ell(\rho_{n})$ given by

$$C\ell\,(X) = P_{spin}(X) \times_{Ad} C\ell(\mathbb{R}^{n}).$$

We say two spinor bundles of X are equivalent iff they are equivalent as bundles of $C\ell(X)$-modules.

A bundle of (real or complex, graded or ungraded) $C\ell(X)$-modules is called irreducible if at each x the fibre is irreducible as a (real or complex, graded or ungraded) module over $C\ell(X_{x})$ [2]. Recall that, when X is a spin manifold ;

(i)the elements of $C\ell\,(X) = P_{spin}(X) \times_{Ad} C\ell(\mathbb{R}^{n})$ are equivalence classes $[(p,a)]$ of pairs (p,a) where $p \in P_{spin}(X)$, $a \in C\ell(\mathbb{R}^{n})$ and $(p,a) \sim (p',a') \Leftrightarrow p' = pu^{-1}$, $a' = uau^{-1}$, *for* some $u \in spin$.

(ii) the elements of $C\ell^{l}{}_{spin}(X)$ are equivalence classes of pairs (p,a) where $p \in P_{spin}(X)$,
$a \in C\ell(\mathbb{R}^{n})$ and $(p,a) \sim (p',a') \Leftrightarrow p' = pu^{-1}$, $a' = au^{-1}$, *for* some $u \in spin$.

(iii) the elements of $C\ell^{r}{}_{spin}(X)$ are equivalence classes of pairs (p,a) where
$p \in P_{spin}(X)$, $a \in C\ell(\mathbb{R}^{n})$ and $(p,a) \sim (p',a') \Leftrightarrow p' = pu^{-1}$, $a' = au^{-1}$, *for* some $u \in spin$.

Proposition 38:

There is a natural pairing $\sec C\ell^{l}{}_{spin}(X) \times \sec C\ell^{r}{}_{spin}(X) = \sec C\ell\,(X)$.

Proof :Given $\alpha \in \sec C\ell^{l}{}_{spin}(X)$, $\beta \in \sec C\ell^{r}{}_{spin}(X)$select representatives (p, a) for $\alpha(x)$ and (p, b) for $\beta(x)$ (with $p \in \pi^{-1}(x)$) and define $(\alpha\beta)(x) \coloneqq [(p, ab)] \in C\ell(X)$. If alternative representatives (pu^{-1}, ua) and (pu^{-1}, bu^{-1}) are chosen for $\alpha(x)$ and $\beta(x)$ we have
$(pu^{-1}, uabu^{-1}) \sim (p, ab)$ and thus $(\alpha\beta)(x)$ is well define element of $C\ell\,(X)$[5].□

Let us now say a word about the $\mathbb{Z}_2$-graded case. There is a natural one-to-one correspondence between classes of bundles of irreducible $\mathbb{Z}_2$-graded modules over $C\ell(X) = C\ell^0(X) \oplus C\ell^1(X)$ and classes of bundles of irreducible modules over $C\ell^0(X)$. Given a bundle $S(X) = S^0(X) \oplus S^1(X)$ of the first kind, $S^0(X)$ is of the second. Given an $S^0(X)$ of the second kind, the bundle $S(X) = C\ell(X) \otimes_{C\ell^0(X)} SO(X)$ is of the first.

Suppose now that $n = 2m$ and $S_{\mathbb{C}}(X)$ is the irreducible complex spinor bundle of X. We shall show explicitly how to split $S_{\mathbb{C}}(X)$ into a direct sum

$S_{\mathbb{C}}(X) \;=\; S_{\mathbb{C}}^{+}(X) \;\oplus\; S_{\mathbb{C}}^{-}(X)$

of $C\ell^0(X)$ -modules. Interpreting $S_{\mathbb{C}}^{+}(X)$ as $S_{\mathbb{C}}^{0}(X)$and $S_{\mathbb{C}}^{-}(X)$as $S_{\mathbb{C}}^{1}(X)$, or the other way around, gives a $\mathbb{Z}_2$-graded module structure to $S_{\mathbb{C}}(X)$.

There is an analogous construction in the real case.

$S(X) = \; S^{+}(X) \oplus S^{-}(X)$ [2].

Recall that every module for $C\ell(\mathbb{R}^n)$ can be written as a direct sum of irreducible ones, and there are at most two equivalence classes of irreducible modules[2].

Proposition 39:

Let $S(X)$ be a real spinor bundle of X. Then $S(X)$ is a bundle of modules over the bundle of algebras $C\ell(X)$. In particular the sections of the spinor bundle are a module over the sections of the Clifford bundle[2,5].

Remark 40: The corresponding fact holds in the complex and $\mathbb{Z}_2$-graded cases and

Sections of $S(X)$ are called spinors.

Definition 41:

The dual spinor bundle $S^*(X)$ is real or complex spinor bundle

$$S^*(X) = \begin{cases} P_{spin}(X) \times_{\mu} L^* \\ P_{spin}(X) \times_{\mu} L_c^* \end{cases}$$

with L^* is a right module of $C\ell_{r,\,s}$ and $\mu{:}Spin(r, s) \to End(L)$ is the representation given by right-multiplication of (inverse)elements in $Spin(r, s)$,and where $L^*_{\mathbb{C}}$ is a complex right module for $C\ell(\mathbb{R}^n)\otimes\mathbb{C}$ and

μ:*Spin*(r, s) → *End*($L_{\mathbb{C}}$) is the representation given by right-multiplication of (inverse) elements in *Spin*(r, s) [5].

Definition 42:

Let U be the universal bundle of the Grassmannian. We call $s^{*}U = S$ the spinor bundle on Q_{2k+1}. Its rank is 2^{k}. We call $s'^{*}U = S'$ and $s''^{*}U \simeq S''$ the two spinor bundles on Q_{2k}, their rank is 2^{k-1}. If f is an automorphism of Q_{2k} that exchanges the two families of k-planes, we have

$$f^{*}S' \simeq S'' \text{ and } f^{*}S'' \simeq S'.$$

It is clear that spinor bundles S on all quadrics Q are homogeneous, i.e., $f^{*}S \simeq S$ $\forall f \in \mathrm{Aut}(Q)_0$, where $\mathrm{Aut}(Q)_0$ is the connected component of the identity in Aut(Q).

From the geometrical description given above, the following theorem is clear.

Theorem43:

(i) Let S', S'' be the spinor bundles on Q_{2k}, and let i:$Q_{2k-1} \to Q_{2k}$ be a smooth hyperplane section. Then $i^{*}S' \simeq i^{*}S'' \simeq S$, where S is the spinor bundle on Q_{2k-1}.

(ii) Let S be the spinor bundle on Q_{2k+i}, and let i:$Q_{2k} \to Q_{2k+i}$ be a smooth hyperplane section. Then $i^{*}S \simeq S' \oplus S''$, where S' and S'' are the spinor bundles on Q_{2k} [4].

Example44: In the definition of spinor bundles, the two embeddings s':$Q_4 \to Gr(l, 3)$ and s'':$Q_4 \to Gr(l, 3)$ are isomorphisms. So the spinor bundles on Q_4 are the universal bundle and the dual of the quotient bundle.

The embedding S: $Q_3 \to Gr(l, 3)$ corresponds to a hyperplane section. If S is the spinor bundle on Q_3, then $S^2S^{*} = TQ_3$. In fact $TQ_4/Q_3 \simeq S^{*} \oplus S^{*} \simeq S^2S^{*} \oplus (1)$ and the exact sequence splits

$$0 \to TQ_3 \to TQ_4/Q_3 \to (1) \to 0$$

On Q_2 the two spinor bundles are the duals of the two line bundles corresponding to two skew-lines on Q_2. On $Q_1 \simeq \mathbb{P}^1$ we can define the spinor bundle to be $\mathcal{O}_{\mathbb{P}^1}(-1)$[4].

Corollary45:

Let $l \subset Q_n$ ($n > 3$) be a line and let S be a spinor bundle on Q_n[4]. Then

$$S|_l = \mathcal{O}_l^{\oplus 2^{\left\lfloor \frac{(n-3)}{2} \right\rfloor}} \oplus \mathcal{O}_l(-1)^{\oplus 2^{\left[\frac{(n-3)}{2} \right]}}.$$

Relations between spinor bundles and spinor structures

For every $x \in M$, the spinor representation $\tau_x: C\ell(g_x) \to End\ \Sigma_x$ defines the real line $a(\tau_x)$ and the circle $c(\tau_x)$ [16]. The set

$$a(\tau) = \bigcup_{x \in M} a(\tau_x) \subset Hom(\Sigma, \bar{\Sigma}^*) = \Sigma^* \otimes \bar{\Sigma}^*.$$

has the structure of a real line bundle over M. The set

$$c(\tau) = \bigcup_{x \in M} c(\tau_x) \subset Hom(\Sigma, \bar{\Sigma}) = \Sigma^* \otimes \bar{\Sigma}.$$

has the structure of a bundle of circles over M, it is a principal $U(1)$-bundle. If this bundle is trivial, i.e; if it has a (global) section $C: M \to c(\tau)$, then we can define the real line bundle over M,

$$b(\tau, C) = \bigcup_{x \in M} b(\tau_x, C(x)) \subset Hom(\Sigma, \Sigma^*) = \Sigma^* \otimes \Sigma^*.$$

Proposition 46.

Let (M, g) be an oriented Riemannian manifold with (V, h) as the local model.

(i) To every spinor bundle there corresponds a Clifford complex (Clifford^c) structure such that the associated spinor bundle is isomorphic to $\tau: C\ell(g) \to End\ \Sigma$[10].

(ii) This Clifford^c structure can be reduced to a spin structure if, and only if, the line bundle above is trivial.

(iii) The Clifford^c structure can be reduced to a Clifford structure if, and only if, the bundle of circles above is trivial.

(iv) If the bundle of circles is trivial, then the resulting Clifford structure can be reduced to a spin structure if, and only if, the real line bundle above is trivial[3].

Definition 47:

Suppose E is a smooth Riemannian vector bundle over a manifold X and that

$\xi: P_{Spin}(E) \to P_{so}(E)$ is a spin structure on E. Then, of course, any connection on $P_{so}(E)$ can be lifted via ξ to a connection on $P_{spin}(E)$, and this, in turn, defines a connection on the associated spinor bundles[2].

We give two equivalent definitions of a connection on a principal G-bundle *with projection* $P \xrightarrow{\pi} X$.

Firstly, we define the vertical tangent space $T_p^v P$ at p as the subspace of $T_p P$ which is tangent to the fiber of the projection $P \xrightarrow{\pi} X$. A connection on $P \overset{\pi}{\rightarrow} X$ is a smoothly varying family of linear subspaces $(T_p^H)_{p \in P}$ of the tangent bundle $T P$ which is everywhere complementary to the vertical distribution $(T_p^v V)_{p \in P}$ and which is invariant under the action of the group G. The distribution $(T_p^H)_{p \in P}$ is called the horizontal distribution. Note that

$TP = T^V P \oplus T^H P$ (12).

Secondly, to give a connection on $P \overset{\pi}{\rightarrow} X$ is to give a connection 1-form ω on P with values in g satisfying two conditions.
It transforms by the adjoint action, i.e. for any g in G, p in P and any γ in $T_p P$, we have

$\omega_{p \cdot g}(\gamma \cdot g) = g^{-1} \omega_p(\gamma) g$.

For any a in g, the associated vector field V_a on P defined by the tangent vector of the curve pe^{ta} at any $p \in P$. It holds that $\omega(V_a) = a$.
Note that the horizontal distribution can be recaptured by taking the kernel of the connection 1-form assigned to it.

Lemma 48:
Given a Euclidean connection on a real vector bundle E, there is a canonical orthogonal connection (i.e. the decomposition $TP = T^V P \oplus T^H P$ is Euclidean) on its orthonormal frame bundle $P \xrightarrow{SO(n)} X$ Reversely, any orthogonal connection on the frame bundle $P \xrightarrow{SO(n)} X$ induces a Euclidean connection on E. Furthermore, these operations are inverse to each other[12].

Lemma 49:
Given a Lie group G and a connection on a principal G-bundle $P \overset{\pi}{\rightarrow} X$ there is an induced connection on any vector bundle $P \times_G V$ coming from a linear representation $G \rightarrow GL(V)$[12].
The curvature of the connection is the g-valued 2-form Ω given by the equation

$$\Omega = d\omega + [\omega, \omega].$$

Example 50: Let $P = P_{so}(E)$ where E is a smooth, oriented Riemannian vector bundle. The Lie algebra SO_n of real, skew-symmetric $n \times n$-matrices. Hence, a connection1-form ω can be considered as an $n \times n$-matrix of 1-forms $\omega = (\omega_{ij})$ where $\omega_{ij} = -\omega_{ji}'$ The corresponding curvature is a matrix of 2-forms $\Omega = \Omega_{ij}$ where

$$\Omega_{ij} = d\omega_{ij} + \sum_{k=1}^{n} \omega_{ik} \Lambda \omega_{kj}$$

Suppose that $\mu =$ $(e_1,\ldots, e_n)$ is just a section of $P_{so}(E)$ over U⊆ X, and it can be lifted to a section $\tilde{\mu}$ of $P_{Spin}(E)$ over U. There are two possible such liftings. They satisfy the relation:

$\xi \circ \tilde{\mu} = \mu$.

The connection 1-form on $P_{SPin}(E)$ is just the lift ξ*ω (the pull down) of the connection 1-form ω on $P_{so}(E)$ [2].

Theorem 51:

Let ω be the connection i-form on $P_{so}(E)$ and let $S(E)$ be any spinor bundle associated to E. Then the covariant derivative ∇^s on $S(E)$ is given locally by the formula [2]

$$\nabla^s = \frac{1}{2} \sum_{i<j} \tilde{\omega}_{ij} \otimes e_i e_j \,.$$

where $\mu =$ $(e_1,\ldots, e_n)$ is a local section of $P_{so}(E)$, $\omega = \mu^*(\omega)$, and where $\Xi = (\sigma_1, \ldots, \sigma_n)$ is a local section of $P_{so}(S(E))$ determined by ω.

Theorem 52:

Let Ω be the curvature 2-form on $P_{so}(E)$ and let $S(E)$ be any spinor bundle associated to E. Then the curvature $\mathcal{R}$ by the formula S of $S(E)$ is given locally

$$R^s = \frac{1}{2} \sum_{i<j} \tilde{\Omega}_{ij} \otimes e_i e_j \,.$$

where $\mu =$ $(e_1,\ldots, e_n)$ is a local section of $P_{so}(E)$, $\tilde{\Omega} = \mu^*(\Omega)$ and where σ is any
section of $S(E)$ [2].

In particular, for any two tangent vectors $\boldsymbol{V}$ and $\boldsymbol{W}$ at $x \in X$, the curvature transformation $\mathcal{R}^s_{V,W}$: $S(E_x) \rightarrow S(E_x)$ is given by the formula

$$R^s_{V,W} = \frac{1}{2} \sum_{i<j} \langle R_{V,W}(e_i, e_j) \rangle e_i e_j \,.$$

where $\mathcal{R}_{v,w}$ is the curvature transformation of E_x.

Definition 53:

Dirac operator D acting on sections of spinor bundle $\Sigma \rightarrow M$ is globally defined as follows.

Let U_ι be an open subset of M and let $e = (e_\mu)_{\mu=1,\ldots,m}$ be a field of (not necessarily orthonormal) frames on U_ι. For every $p \in U_\iota$, the components of the metric tensor g with respect to e at p are $g_{\mu\nu}(p) = g(e_\mu(p), e_\nu(p))$ and there is the inverse $g^{\mu\nu}(p)$ of $g_{\mu\nu}(p)$. The restriction of the Dirac operator to U_ι is[2],

$$D = g^{\mu\nu}\tau(e_\mu)\nabla_{e_\nu}.$$

The Dirac operator on M well defined by its restrictions to the sets (U) providing an open cover of M.

Note that the connection ∇^{TM} giving rise to the connection on the spinor bundle is

required to be metric, but may have torsion. In fact, there are mathematical and physical reasons to consider metric connections with torsion in the context of spinors .

Definition 54:

A Dirac bundle over a Riemannian manifold X is a bundle S of left modules over $C\ell(X)$ together with a Riemannian metric and connection on S having properties

$\langle e_1\sigma_1, e_2\sigma_2\rangle = \langle \sigma_1, \sigma_2\rangle$ and $\nabla(\varphi\sigma) = (\nabla\varphi)\sigma + \varphi(\nabla\sigma)$, for all $\varphi \in \Gamma\big(C\ell(X)\big)$,

$\sigma \in \Gamma(S)$.

The operator D is elliptic if the linear map $\sigma_\xi(D)$: $E_x \to E_x$ is an isomorphism for all

$\xi \neq 0$

Lemma 55:

Let D be the Dirac operator of the bundle S defined above. Then for any $\xi \in T^*(X) \cong T(X)$ we have that

$$\sigma_\xi(D) = i\xi$$
$$\sigma_\xi(D^2) = \|\xi\|^2$$

where the symbol on the right denotes Clifford multiplication by the vector ξ and the scalar $\|\xi\|^2$. In particular, both D and D^2 are elliptic operators[2].

Theorem 56:

Let X be a complete Riemannian manifold and let D be the Dirac operator of any Dirac bundle S over X. Then the closure of D in $L^2(S)$ is a self-adjoint operator. Furthermore, $\ker(D) = \ker(D^2)$ on $L^2(S)$. Where L a canonical bundle map L: $C\ell(X) \to C\ell(X)$ defines by $L(\varphi) = -\sum e_j \varphi e_j$ [2].

Lemma 57:

Let D, S and X be as above. Then for any $f \in C^{\infty}(X)$ and any $\varphi \in \Gamma(S) \subset C^{\infty}(X)$,

we have that

$D(f\varphi) = (\text{grad } f) \,.\, \varphi + fD\varphi$.

Proof: $D(f\varphi) = \sum e_j . \nabla e_j (f\varphi) = \sum e_j \{(e_j f)\varphi + f\nabla e_j \varphi\} = \sum (e_j f)\, e_j . \varphi) + fD\varphi) = (\text{grad } f) \,.\, \varphi + fD\varphi$ [2].□

Having discussed Dirac bundles in general terms, it is now time to look hard at some important examples. We begin with the basic ones.

Example 58: (an historical case) Let $X = R^n$, Euclidean n-space, and let $S = R^n \times V$ where V is some finite dimensional module for $C\ell_n$. In this case the Dirac operator is a constant coefficient operator (on V-valued functions) of the form[2]

$$D = \sum_{K=1}^{n} \gamma_k \frac{\partial}{\partial x_k}$$

where each γ_k is a linear map $\gamma_k : V \rightarrow V$ and where

$\gamma_j \gamma_k + \gamma_k \gamma_j = -2\delta_{jk}$.

This particular operator has historical roots in physics. In the 1920s(2), the physicist P.A.M. Dirac was searching for a Lorentz-invariant first-order differential operator whose square would be the Klein-Gordon operator. Thus he was essentially led to search for a first order operator D of the form above which satisfied the equation $D^2 = \Delta$ where $\Delta = -\sum \partial^2/\partial x^2$ is the positive Laplacian in $\mathbb{R}^n$. Realizing that the γ_k's must be matrices, he was led immediately by this equation to the above relations, which we recognize now as the generating relations of a representation of $C\ell_n$.

Let $n = 1$, so that $C\ell_1 = V = \mathbb{C} \cong \mathbb{R}^2$. Then we have [10]

$$D = i\frac{\partial}{\partial x_1}$$

the generator of a basic semi-group of unitary operators on L^2.

Let $n = 2$, so that $C\ell_2 = V = \mathbb{H} \cong \mathbb{C} \otimes \mathbb{C}$. The decomposition of $\mathbb{H}$ into $\mathbb{C} \otimes \mathbb{C}$ is natural and corresponds to the $\mathbb{Z}_2$-grading $C\ell_2^0 \otimes C\ell_2^1$ [10]

$$D = e_1\frac{\partial}{\partial x_1} + e_2\frac{\partial}{\partial x_2}$$

has the form $D = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} \\ \frac{\partial}{\partial \overline{z}} & 0 \end{pmatrix}$ where $\partial/\partial\overline{z} = \partial/\partial x_1 + \mathrm{i}\,\partial/\partial x_2$

Let n = 3, so that $C\ell_3 = \mathbb{H} \otimes \mathbb{H}$ and $V = \mathbb{H}$. $C\ell_3$ has two representations on $\mathbb{H}$ given as identify $\mathbb{R}^3$ with Im($\mathbb{H}$), by letting i,j,k act on either the right or the left in $\mathbb{H}$. on the left, we get Dirac operator on $\mathbb{H}$

$$D = i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}$$

Let n = 4, so that $C\ell_4 = \mathbb{H}(2)$ and $V = \mathbb{H}^2 = \mathbb{H} \otimes \mathbb{H}$. Here again the splitting corresponds to a $\mathbb{Z}_2$-grading of the module V, and D interchanges parts. To describe the full Dirac operator we consider first the following $\mathbb{H}$ under the standard basis $(1,i,j,k)$

$$\frac{\partial}{\partial\overline{q}} = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}, \frac{\partial}{\partial q} = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}$$

then $$D = \begin{pmatrix} 0 & -\frac{\partial}{\partial q} \\ \frac{\partial}{\partial \overline{q}} & 0 \end{pmatrix}$$

Thus, left multiplication be represented by complex 2×2-matrices σ_0, σ_2 and σ_3 respectively, then the operator $\partial/\partial\overline{q}$ becomes

$$\frac{\partial}{\partial\overline{q}} = \frac{\partial}{\partial x_0} + \sigma_1\frac{\partial}{\partial x_1} + \sigma_2\frac{\partial}{\partial x_2} + \sigma_3\frac{\partial}{\partial x_3}.$$

The matrices σ_k can be chosen to be the classical *Pauli matrices*:

$$\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Note that these matrices generate the fundamental representation of $C\ell_3$ in complex form[2,10].

Example 59: (*the Clifford bundle*). Let $S = C\ell(X)$ with its canonical Riemannian connection, and view $C\ell(X)$ as a bundle of left modules over itself by left Clifford multiplication. The Dirac operator in this case is a square root of the classical Hodge Laplacian[2].

Example 60: (*the spinor bundles*). Suppose X is a spin manifold with a spin structure on its tangent bundle. Let S be any spinor bundle associated to $T(X)$. Then S is a bundle of modules over $C\ell(X)$, and S carries a canonical Riemannian connection which has property of definition54. The Dirac operator in this case was first written down by Atiyah and Singer in their work on the Index theorem. Finding this operator was a major accomplishment, and for this reason we shall call it the Atiyah-Singer operator[2].

Notation. For spin manifolds X of even dimension we shall denote the (unique) irreducible complex spinor bundle by $\$_{\mathbb{C}}$; and when dim($X$) $\not\equiv$ 3(mod4), we denote the

irreducible real spinor bundle by \$. In both cases the Atiyah-Singer operator will be written $\not{D}$.

These basic examples each generate large families of new examples by the following construction.

Let S and E be a given Dirac bundle with connection∇^{s} and ∇^{E} over a Riemannian manifold X. Then the tensor product $S \otimes E$ is again a bundle of left modules over $C\ell(X)$, where for $\varphi \epsilon C\ell(X)$, $\sigma \in S$, $e \in E$, $\varphi.(\sigma \otimes e) = (\varphi.\sigma) \otimes e$[2].

Furthermore, we can equip $S \otimes E$ with the canonical tensor product connection,

$\nabla = \nabla^{s} \otimes \nabla^{E}$ which is defined on sections of the form $\sigma \otimes e$ by the formula

$\nabla(\sigma \otimes e) = (\nabla^{s}\sigma) \otimes e + \sigma \otimes (\nabla^{E} e)$.

CONCLUSION

The researchs have discussed how Riemannian geometry, including the theory of

spinor fiber bundle, fits into the general setting of principal fibre bundles. We see that the spinor bundles defined as vector bundles whose fibers carry spinor representations of the Clifford algebras $C\ell(g)$, spinor fields are sections of spinor bundles. As it turns out, the theory of spinor structures for Riemannian geometry extends naturally to the general setting, and a large part of the work here has been in establishing the appropriate classifications for abstract spinor structures.

With this constructive classification tool, we have investigated the usefulness of spinor structures of Clifford algebra. We have also given an explicit description of an important physical applications of spinor bundle, and used a spinor connection and the constant Dirac matrices of special relativity to define the Dirac operator using the Clifford algebra of spacetime and Dirac operation.

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